

Rings, Group Rings, and Their Graphs

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Abstract

We associate some graphs to a ring R and we investigate the interplay between the ring-theoretic properties of R and the graph-theoretic properties of the graphs associated to R . Let $Z(R)$ be the set of zero-divisors of R . We define an undirected graph $\Gamma(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, where distinct vertices x and y are adjacent if $xy = 0$ or $yx = 0$. We investigate Isomorphism Problem for zero-divisor graphs of group rings RG . Let S_k denote the sphere with k handles, where k is a non-negative integer, that is, S_k is an oriented surface of genus k . The genus of a graph is the minimal integer n such that the graph can be embedded in S_n . The annihilating-ideal graph of R is defined as the graph $\mathbb{A}\mathbb{G}(R)$ with vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ (the set of ideals with non-zero annihilators) such that two distinct vertices I and J are adjacent if $IJ = (0)$. We characterize Artinian rings whose annihilating-ideal graphs have finite genus. Finally, we extend the definition of the annihilating-ideal graph to non-commutative rings.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

The study of algebraic structures using the properties of graphs became an exciting research topic in the last twenty years, leading to many fascinating results and new research problems. Here we associate some graphs to a ring R to study the structure of zero-divisors in R as well as the structure of annihilating-ideal graph of R .

Let us start with zero-divisor graph. Let R be a commutative ring with $1 \neq 0$. We associate a (simple) graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R , and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. The concept of a zero-divisor graph of a commutative ring was introduced by I. Beck in [14], which was mainly concerned with colorings of rings. The definition given

above differs from that in the earlier work of D. D. Anderson and M. Naseer [12] and Beck in since 0 is not taken as a vertex of $\Gamma(R)$. In [35], Redmond introduced the notion of an ideal based zero-divisor graph, generalizing the notion of $\Gamma(R)$ and this idea is pursued further in [28].

D. F. Anderson and P. S. Livingston [9] gave several fundamental results concerning $\Gamma(R)$ for a commutative ring R using the above definition. Also, the next two examples show that non-isomorphic rings may have isomorphic zero-divisor graphs. Figures 1.1 and 1.2 appeared in [9].

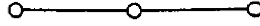


Figure 1.1: Zero-divisor graphs for $\mathbb{Z}_6, \mathbb{Z}_8$ and $\mathbb{Z}_2[X]/(X^3)$.

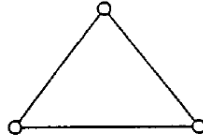


Figure 1.2: Zero-divisor graphs for $\mathbb{F}_4[X]/(X^2)$ and $\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$.

Let S_i denote the sphere with i handles, where i is a non-negative integer. That is, S_i is an orientable surface of genus i . The genus of a graph Γ , denoted $\gamma(\Gamma)$, is the minimal integer m such that the graph can be embedded in S_m . Intuitively, Γ is embedded in a surface if it can be drawn in the surface so

that its edges intersect only at their common vertices. For vertices x and y of Γ , let $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of Γ is $\text{diam}(\Gamma) = \sup\{d(x, y) | x \text{ and } y \text{ are vertices of } \Gamma\}$. The girth of Γ , denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle in Γ ($\text{gr}(\Gamma) = \infty$ if Γ contains no cycles).

In [10], the following question was asked: For which finite commutative ring R is $\Gamma(R)$ planar (a graph which has genus 0)? In [3] a partial answer was given, but the question was not answered for local rings of order 32. In [39], and independently in both [41] and [17], it was shown that no local ring of order 32 has a planar zero-divisor graph. Furthermore, Smith [39] gave a complete list of finite planar rings. In [43], a complete list of finite rings whose zero-divisor graphs have genus 1 was given. Also, Wickham in [44] showed that there are finitely many finite commutative rings with finite positive genus zero-divisor graphs. In [7], Aliniaefard and Behboodi generalized the Wickham's Theorem and showed that there are finitely many (not necessarily finite) commutative rings with finite positive genus zero-divisor graphs.

Now, we introduce annihilating-ideal graph of a commutative ring R . Let $\mathbb{A}(R)$ be the set of ideals with non-zero annihilators. The annihilating-ideal graph of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}(R)^* = \mathbb{A} \setminus \{(0)\}$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Thus $\mathbb{AG}(R)$ is the empty graph if and only if R is an integral domain. This notion of annihilating-ideal graph was first introduced and systematically studied by Behboodi and Rakeei in [15, 16]. Recently it has received a great

deal of attention from several authors, for instance, see [1, 2] and [6].

In Chapter 1, we present some basic concepts, definitions, and facts in the topological graph theory, ring and group ring theory, which are needed in the sequel.

In Chapter 2, we first introduce the zero-divisor graphs for commutative rings and present some preliminary results. Then we investigate the interplay between the ring-theoretic properties of group ring RG and the graph-theoretic properties of $\Gamma(RG)$. We characterize finite abelian group rings RG for which either $\text{diam}(\Gamma(KG)) \leq 2$ or $gr(\Gamma(RG)) \geq 4$. Also, the isomorphism problem for zero-divisor graphs of group rings is studied. First, it is shown that two finite semisimple group rings are isomorphic if and only if their zero-divisor graphs are isomorphic. Also, it is shown that rank and cardinality of a finite abelian p -group is determined by the zero-divisor graph of its modular group ring. Finally, we show that finite non-commutative reversible group rings with commutative coefficient rings are determined by their zero-divisor graphs.

A few preliminary results in annihilating-ideal graphs are given in Chapter 3. It is shown that if R is an Artinian ring such that $0 < \gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then R has only finitely many ideals, extending a recent result in [6].

In Chapter 4, we extend the definition of the annihilating-ideal graph to non-commutative rings. We introduce various ways to define the annihilating-ideal graph of a non-commutative ring. The first definition gives a directed graph denote by $(APO)G(R)$. The other definition yield an undirected

graph denoted by $\overline{(APO)}G(R)$. It is shown that $(APO)G(R)$ is not connected but $\overline{(APO)}G(R)$ is connected and the diameter of $\overline{(APO)}G(R)$ is less than or equal to 3. Also, we show that if $(APO)G(R)$ has DCC (resp., ACC) on its vertices, then R is an Artinian (resp., Noetherian) ring. It is shown that $\overline{(APO)}G(R)$ has some features similar to that of an annihilating-ideal graph. Finally, we investigate the diameter and girth of square matrices over commutative rings $M_{n \times n}(R)$, when $n \geq 2$. It is shown that $diam(\overline{(APO)}G(M_{n \times n}(R))) \geq 2$ and $g(\overline{(APO)}G(M_{n \times n}(R))) = 3$, where $n \geq 2$.

1.2 Topological Graph Theory

The topological graph theory is a branch of graph theory, which studies the embedding of graphs in surfaces, spatial embeddings of graphs, and graphs as topological spaces. It also studies immersions of graphs. Embedding a graph in a surface means that we want to draw the graph on a surface, a sphere for example, without two edges intersecting. In this section we present some basic concepts and definitions in the topological graph theory, which are needed in the sequel.

What is a Graph? A graph Γ is a triple consisting of a vertex set $V(\Gamma)$, an edge set $E(\Gamma)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. A subgraph of a graph Γ is a graph θ such that $V(\theta) \subseteq V(\Gamma)$ and $E(\theta) \subseteq E(\Gamma)$ and the assignment of end-

points to edges in θ is the same as in Γ . We then write $\theta \subseteq \Gamma$ and say that " Γ contains θ ". A graph is finite if its vertex set and edge set are finite. A loop is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints. A simple graph is a graph having no loops or multiple edges. For a graph Γ , the degree of a vertex v of Γ , denoted by $\deg(v)$, is the number of edges of Γ incident with v and $m(\Gamma) = \max\{\deg(v) : v \in V(\Gamma)\}$.

Path and Cycle A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A graph G is connected if each pair of vertices in G belongs to a path; otherwise, G is disconnected. A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The path and cycle with n vertices are denoted P_n and C_n , respectively; an n -cycle is a cycle with n vertices.

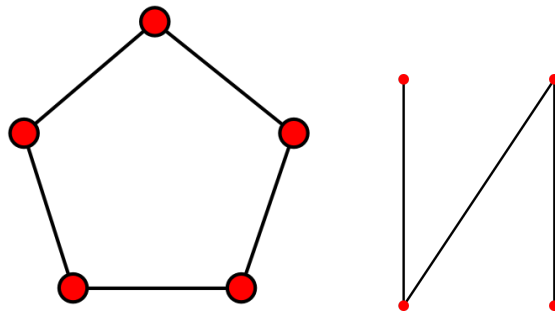


Figure 1.3: C_5 and P_4

Complete Graph and Complete Bipartite Graph A complete graph is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted K_n . A graph Γ is bipartite if $V(\Gamma)$ is the union of two disjoint (possibly empty) independent sets called partite sets of Γ . A complete bipartite graph or biclique is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the partite sets have sizes r and s , the complete bipartite graph is denoted $K_{r,s}$.

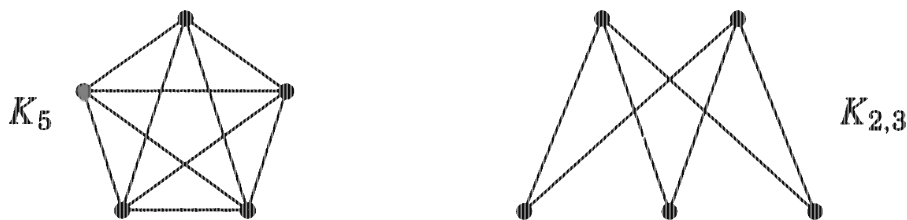


Figure 1.4: K_5 and $K_{2,3}$

Diameter and Girth of a Graph For vertices x and y of Γ , let $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such a path). The diameter of Γ is $\text{diam}(\Gamma) = \sup\{d(x, y) | x \text{ and } y \text{ are vertices of } \Gamma\}$. The girth of Γ , denoted by $gr(\Gamma)$, is the length of a shortest cycle in Γ ($gr(\Gamma) = \infty$ if Γ contains no cycles).

Isomorphism Between two simple Graphs An isomorphism from a simple graph Γ to a simple graph θ is a bijection $f : V(\Gamma) \rightarrow V(\theta)$ such that $uv \in E(\Gamma)$ if and only if $f(u)f(v) \in E(\theta)$. We say " Γ is isomorphic to θ ", written $\Gamma \cong \theta$, if there is an isomorphism from Γ to θ .

Neighborhoods and Topological Space Let X be a set; the elements of X are usually called points, though they can be any mathematical object. We allow X to be empty. Let N be a function assigning to each x in X a non-empty set $N(x)$ of subsets of X . The elements of $N(x)$ will be called neighborhoods of x with respect to N (or, simply, neighborhoods of x). The function N is called a neighborhood topology if the axioms below are satisfied (X with N is called a topological space):

(1) If N is a neighborhood of x , then $x \in N$. This means every point belongs to every neighborhood of that point.

(2) If N is a subset of X containing a neighborhood of x , then N is a neighborhood of x . This means that every superset of a neighborhood of a point x in X is again a neighborhood of x .

(3) The intersection of two neighborhoods of x is a neighborhood of x .

(4) Any neighborhood N of x contains a neighborhood M of x such that N is a neighborhood of each point of M .

A standard example of such a system of neighborhoods is for the real line R , where a subset N of R is defined to be a neighborhood of a real number x if there is an open interval containing x and contained in N .

A subset U of X to be open if U is a neighborhood of all points in U . It is a remarkable fact that the open sets satisfy the elegant axioms given below, and that, given these axioms, we can recover the neighborhoods satisfying the above axioms by defining N to be a neighborhood of x if N contains an open set U such that $x \in U$.

A topological space is then a set X together with a collection of subsets of X , called open sets and satisfying the following axioms:

- (1) The empty set and X itself are open.
- (2) Any union of open sets is open.
- (3) The intersection of any finite number of open sets is open.

The collection T of open sets is then also called a topology on X , or, if more precision is needed, an open set topology. The sets in T are called the open sets, and their complements in X are called closed sets. A subset of X may be neither closed nor open, either closed or open, or both.

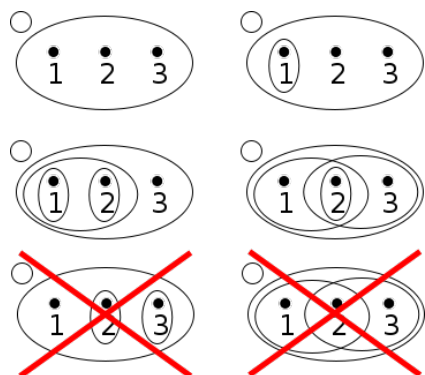


Figure 1.5: Four examples and two non-examples of topologies on the three-point set $\{1, 2, 3\}$. The bottom-left example is not a topology because the union of $\{2\}$ and $\{3\}$ [i.e., $\{2, 3\}$] is missing; the bottom-right example is not a topology because the intersection of $\{1, 2\}$ and $\{2, 3\}$ [i.e., $\{2\}$], is missing.

Homeomorphic Topological Spaces A function $f : X \rightarrow Y$ between topological spaces X and Y , is called continuous if for each $x \in X$ and each neighbourhood N of $f(x)$ there is a neighbourhood M of x such that $f(M) \subseteq N$. This relates easily to the usual definition in analysis. Equivalently, f is continuous if the inverse image of every open set is open. This is an attempt to capture the intuition that there are no "jumps" or "separations" in the function. A homeomorphism is a bijection that is continuous and whose inverse is also continuous. Two spaces are called homeomorphic if there exists a homeomorphism between them. From the standpoint of topology, homeomorphic spaces are essentially identical.

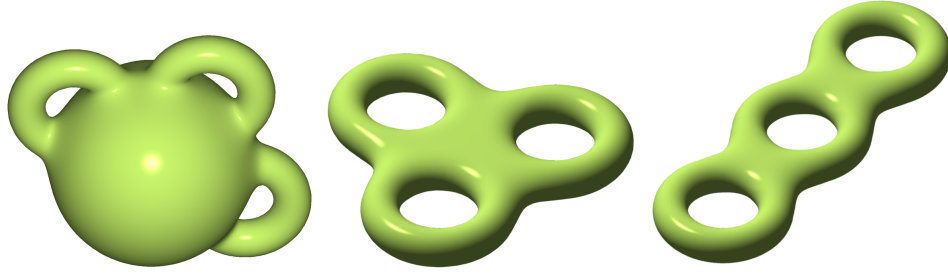


Figure 1.6: Three homeomorphic spaces.

Connected Sum of 2-manifolds A topological space X is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be connected. A subset of a topological space is said to be connected if it is connected under its subspace topology. A 2-manifold is a topological space that each point has a neighbourhood where is homeomorphic to $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 < 1\}$. A connected sum of two 2-manifolds is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres. The operation of connected sum is denoted by $\#$, for example $A \# B$ denotes the connected sum of A and B .

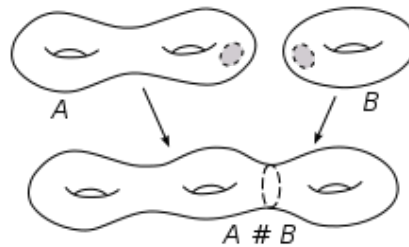


Figure 1.7: Illustration of connected sum.

Embedding a Graph in a Topological space Let Γ be a graph, with $V(\Gamma) = \{v_1, \dots, v_n\}$ and $E(\Gamma) = \{x_1, \dots, x_m\}$. Let M be a 2-manifold. An embedding of Γ in M is a subspace $S(M)$ of M such that,

$$S(M) = \left(\bigcup_{i=1}^n v_i(M)\right) \cup \left(\bigcup_{j=1}^m x_j(M)\right),$$

Where

- (1) $v_1(M), \dots, v_n(M)$ are distinct points of M ,
- (2) $x_1(M), \dots, x_m(M)$ are mutually disjoint arcs (an arc in M is a homeomorphic image of $S = \{(x, y) \in R \times R \mid x^2 + y^2 = 1\}$ in M ,
- (3) $x_j(M) \cap v_i(M) = \emptyset$, $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$,
- (4) if $x_j = (v_{j1}, v_{j2})$, then the open arc $x_j(M)$ has $v_{j1}(M)$ and $v_{j2}(M)$ as end points; $j = 1, \dots, m$.

Intuitively, Γ is embedded in a topological space if it can be drawn in the topological space so that its edges intersect only at their common vertices.

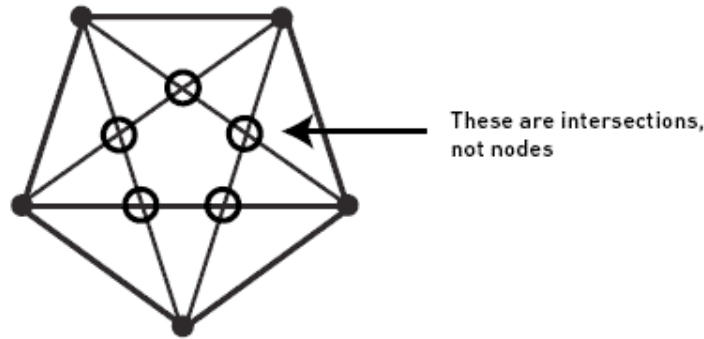


Figure 1.8: K_5 can not be embedded on a plane (see [18, Corollary 4.2.11]).

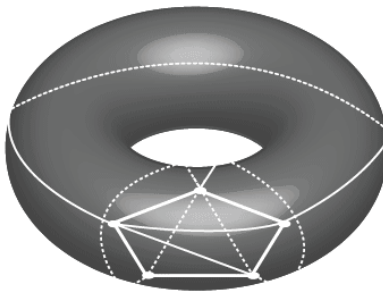


Figure 1.9: K_5 can be embedded in a torus.

Genus of a Graph A torus is a topological space homeomorphic to $S \times S$, where $S = \{(x, y) \in R \times R \mid x^2 + y^2 = 1\}$.

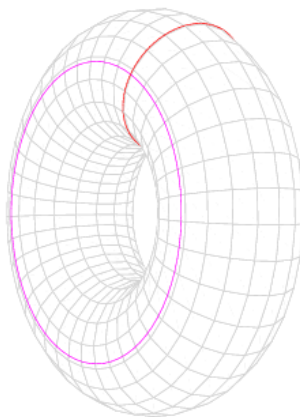


Figure 1.10: A torus

Let S_i denote the connected sum of a sphere with i tori, where i is a nonnegative integer. The genus of a graph Γ , denoted $\gamma(\Gamma)$, is the minimal integer m such that the graph can be embedded in S_m . For example, from

Figures 1.8 and 1.9, $\gamma(K_5) = 1$, i.e., K_5 has genus 1.

Theorem 1.2.1 (See [42, Page 68])

$$(1.1) \quad \gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \quad \text{for all } n \geq 3.$$

$$(1.2) \quad \gamma(K_{m,n}) = \left\lceil \frac{(n-2)(m-2)}{4} \right\rceil \quad \text{for all } n \geq 2 \text{ and } m \geq 2.$$

1.3 Ring Theory and Group Ring Theory

We now recall some of concepts and definitions in ring theory and group ring theory.

1.3.1 Ring Theory

We present some basic facts and well-known theorems in ring theory which will be used frequently. Throughout this thesis all rings are associative with identity.

Definition 1.3.1 *Let X be a subset of R . The left annihilator of X is the set*

$$Ann_l(X) = \{a \in R : ax = 0, \forall x \in X\}.$$

Similarly, we define the right annihilator of X by:

$$Ann_r(X) = \{a \in R : xa = 0, \forall x \in X\}.$$

Lemma 1.3.2 *Let R be a ring and $x \in R$. Then $Rx \cong R/Ann_l(x)$ and $xR \cong R/Ann_r(x)$.*

Proof. Define $f : R \rightarrow Rx$, by $f(r) = rx$. Then by the first isomorphism Theorem $Rx \cong R/Ann_l(x)$. Similarly, $xR \cong R/Ann_r(x)$.

Artinian and Noetherian Rings

Let R be a ring. An R -module N is left Noetherian, named after Emmy Noether, if it satisfies the ascending chain condition (ACC) on left R -submodules of N , right Noetherian if it satisfies the ascending chain condition on right R -submodules, and Noetherian or two-sided Noetherian if it is both left and right Noetherian.

A ring R is left Noetherian if it is a left Noetherian R -module, right Noetherian if it is a right Noetherian R -module, and Noetherian or two-sided Noetherian if it is both left and right Noetherian. For commutative rings the left and right definitions coincide, but in general they are different from each other. The following conditions are equivalent:

1. R is left Noetherian.
2. Every left ideal I in R is finitely generated.
3. Every non-empty set of left ideals of R , partially ordered by inclusion, has a maximal element with respect to set inclusion.

Similar results hold for right Noetherian rings.

An R -module M is left Artinian if it satisfies the descending chain condition on left R -submodules of M , right Artinian if it satisfies the descending

chain condition (DCC) on right R -submodules of M , and Artinian or two-sided Artinian if it is both left and right Artinian.

A ring R is left Artinian if it is a left Artinian R -module, right Artinian if it is a right Artinian R -module, and Artinian or two-sided Artinian if it is both left and right Artinian. For commutative rings the left and right definitions coincide, but in general they are different from each other. Artinian rings are named after Emil Artin. Also, the definition for a ring R to be left Artinian is equivalent to saying that every non-empty set of left ideals of R , partially ordered by inclusion, has a minimal element with respect to set inclusion.

The similar result holds for right Artinian rings.

In 1939, Hopkins and Levitzki independently discovered that the *DCC* is actually a stronger condition than the *ACC*. Levitzki proved that all right Artinian rings with identity are right Noetherian [25], while Hopkins showed the same result holds for left Artinian rings and left Noetherian [20]. These results together give that all Artinian rings with identity are Noetherian and are summarized in the following theorem, named after both mathematicians.

Theorem 1.3.3 (Hopkins-Levitzki)[25] *Let R be a right (left) Artinian ring with identity. Then R is right (left) Noetherian.*

Local Rings

A ring R is a local ring if it has a unique maximal ideal. A Noetherian local ring (R, \mathfrak{m}) is called *Gorenstein* if $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) = 1$.

Theorem 1.3.4 [13, Theorems 8.7] *Let R be a commutative Artinian ring. Then $R \cong R_1 \times \cdots \times R_n$, where each R_i ($1 \leq i \leq n$) is a local commutative Artinian ring.*

The Jacobson radical of ring R , denote by $J(R)$, is the intersection of all of the maximal left ideals of R .

Nakayama's Lemma

Nakayama's Lemma also known as the Krull-Azumaya Theorem [13], governs the interaction between the Jacobson radical of a ring and its finitely generated modules.

Theorem 1.3.5 (Nakayama's Lemma)[24, (4.22)] *Let R be a ring. For any left ideal $I \subseteq R$, the following statements are equivalent:*

- (1) $I \subseteq J(R)$.
- (2) *For any finitely generated left R -module M , $IM = M$, implies that $M = 0$.*
- (3) *For any left R -submodule $N \subseteq M$ such that M/N is finitely generated, $N + IM = M$ implies that $M = N$.*

1.3.2 Group Ring Theory

A group ring is a free module and at the same time a ring, constructed in a natural way from any given ring and any given group.

Let G be a group (not necessarily finite) and R a ring. We denote by RG the set of all formal linear combinations of the form

$$\alpha = \sum_{g \in G} a_g g,$$

where $a_g \in R$ and $a_g = 0$ almost everywhere, that is, only a finite number of coefficients are different from 0 in each of these sums. As a matter of fact, all sums considered in this thesis will be finite in this sense, even when the summation index runs over an infinite set. It should be understood that we are always making this assumption, even though we shall not state it explicitly again.

Notice that it follows from our definition that given two elements, $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g \in RG$, we have that $\alpha = \beta$ if and only if $a_g = b_g$, $\forall g \in G$.

We define the sum of two elements in RG componentwise:

$$\left(\sum_{g \in G} a_g g\right) + \left(\sum_{g \in G} b_g g\right) = \sum_{g \in G} (a_g + b_g) g$$

.

Also, given two elements $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g \in RG$ we define their product by

$$\alpha\beta = \sum_{g, h \in G} a_g b_h gh.$$

It is easy to verify that, with the operations above, RG is a ring, which has a unity; namely, the element $1 = \sum_{g \in G} u_g g$, where the coefficient corresponding to the unit element of the group is equal to 1 and $u_g = 0$ for every other element $g \in G$.

We can also define a product of elements in RG by elements $\lambda \in R$ as

$$\lambda \left(\sum_{g \in G} a_g g \right) = \sum_{g \in G} (\lambda a_g) g.$$

The set RG , with the operations defined above, is called the group ring of G over R . In the case where R is commutative, RG is also called the group algebra of G over R .

Definition 1.3.6 Augmentation mapping *The homomorphism $\varepsilon : RG \rightarrow R$ given by*

$$\varepsilon \left(\sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g$$

*is called the **augmentation mapping** of RG and its kernel, denoted by $\Delta(G)$, is called the **augmentation ideal** of RG .*

Notice that if an element $\alpha = \sum_{g \in G} a_g g$ belong to $\Delta(G)$, then $\varepsilon(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g = 0$. So, we can write α in the form:

$$\alpha = \sum_{g \in G} a_g g - \sum_{g \in G} a_g = \sum_{g \in G} a_g (g - 1).$$

Therefore, the set $\{g - 1 : g \in G, g \neq 1\}$ is a basis of $\Delta(G)$ over R , and we can write

$$\Delta(G) = \left\{ \sum_{g \in G} a_g(g - 1) : g \in G, g \neq 1, a_g \in R \right\}.$$

Definition 1.3.7 For a subgroup H of G , we shall denote by $\Delta_R(G, H)$ the left ideal of RG generated by the set $\{h - 1 : h \in H\}$. That is,

$$\Delta(G, H) = \left\{ \sum_{h \in H} \alpha_h(h - 1) : \alpha_h \in RG \right\}.$$

if H is normal, then $\Delta(G, H)$ is a 2-sided ideal and $R(G/H) \cong RG/\Delta(G, H)$ While working with a fixed ring R we shall omit the subscript and denote this ideal simply by $\Delta(G, H)$. Note that the ideal $\Delta(G, G)$ coincides with the ideal $\Delta(G)$.

Given a group ring RG and a finite subset X of the group G , we denote by \hat{X} the following element of RG :

$$\hat{X} = \sum_{x \in X} x.$$

Note that, if G is a finite group (in particular if $X = G$), then $\hat{G} = \sum_{g \in G} g$ and $R\hat{G} = \{\sum_{i=1}^n r_i \hat{G} : r_i \in R \text{ for all } i\} = \{r\hat{G} : r \in R\}$.

Notation. For any subset Y , we denote by $|Y|$ the cardinality of Y .

Lemma 1.3.8 [29, Lemma 3.4.3] *Let H be a subgroup of a group G and let R be a ring. Then $\text{Ann}_r(\Delta(G, H)) \neq 0$ if and only if H is finite. In this*

case, we have

$$\text{Ann}_r(\Delta(G, H)) = \hat{H}(RG).$$

Furthermore, if H is normal in G . Also, the element \hat{H} is central in RG and we have

$$\text{Ann}_r(\Delta(G, H)) = \text{Ann}_l(\Delta(G, H)) = \hat{H}(RG).$$

In particular, if G is finite,

$$\text{Ann}_r(\Delta(G)) = \text{Ann}_l(\Delta(G)) = R\hat{G}.$$

Now, we present the following theorem which will be used frequently in this thesis.

Theorem 1.3.9 (Perlis-Walker)[29, Theorem 3.5.4] *Let G be a finite abelian group, of order n , and let K be a field such that $\text{Char}(K) \nmid n$. Then $KG \cong \bigoplus_{d|n} a_d K(\zeta_d)$ where ζ_d is a primitive root of unity of order d and $a_d = \frac{n_d}{[K(\zeta_d):K]}$ with n_d denoting the number of elements of order d in G .*

Isomorphism Problem for Group rings

The isomorphism problem of group rings appears for the first time, in regard to integral group rings, in G. Higman's P.h.D Thesis [21]. It was first posed as a problem in the Algebra Conference at Michigan in 1947 by T. M. Thrall,

who formulated it in the following terms:

”Given a group G and a field K , determine all groups H such that $KG \cong KH$.”

A special case of the isomorphism problem is when the coefficient ring is a field of order p , which is called the modular isomorphism problem. Let F be a field of order p . The modular isomorphism problem asks if the following is true?

Let P and Q be finite p – groups. Then $FP \cong FQ \Rightarrow P \cong Q$.

Only a few positive results are known regarding this problem. In general, the question remains open.

Chapter 2

Zero-Divisor Graphs for Group Rings

In this chapter, we first introduce zero-divisor graphs for commutative rings and present some preliminary results. Then we investigate the interplay between the ring-theoretic properties of group rings RG and the graph-theoretic properties of their zero-divisor graphs $\Gamma(RG)$. Finite commutative group rings RG for which either $\text{diam}(\Gamma(RG)) \leq 2$ or $\text{gr}(\Gamma(RG)) \geq 4$ are characterized. Next we investigate the isomorphism problem for zero-divisor graphs of group rings. First, it is shown that rank and the cardinality of a finite abelian p -group are determined by the zero-divisor graph of its modular group ring. It is also shown that two finite semisimple group rings are isomorphic if and only if their zero-divisor graphs are isomorphic. Finally, we show that finite noncommutative reversible group rings are determined by their zero-divisor

graphs.

2.1 Preliminaries

Let R be a commutative ring with 1 and let $Z(R)$ be the set of zero-divisors of R . We associate a simple graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R , and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain. The main object of the investigation of zero-divisor graphs is to study the interplay of ring-theoretic properties of R with graph-theoretic properties of $\Gamma(R)$. This study helps illuminate the structure of $Z(R)$. The notion of a zero-divisor graph of a commutative ring was introduced by I. Beck in [14], where he was mainly interested in colorings. This investigation of colorings of a commutative ring was then continued by D. D. Anderson and M. Naseer in [12]. Their definition was slightly different from ours; they let all elements of R be vertices and had distinct x and y adjacent if and only if $xy = 0$.

Example 2.1.1 (a) *The vertex set for $\Gamma(\mathbb{Z}_6)$ is $\{2, 3, 4\}$ and the zero divisor graph is shown in Figure 2.1.*

(b) *The vertex set for $\Gamma(\mathbb{Z}_{16})$ is $\{2, 4, 6, 8, 10, 12, 14\}$ and the zero divisor graph is shown in Figure 2.2.*

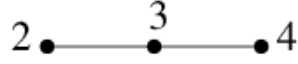


Figure 2.1: The zero-divisor graph of \mathbb{Z}_6 .

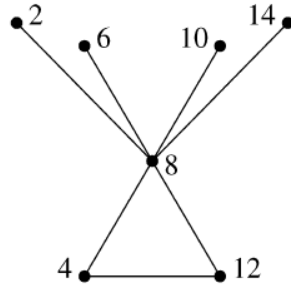


Figure 2.2: The zero-divisor graph of \mathbb{Z}_{16} .

Example 2.1.2 Let $R = \mathbb{Z}_8[X]/(2X, X^2)$, and let x denote the image of X in R . Then the vertex set for $\Gamma(R)$ is the set $(2, x)^* = \{2, 4, 6, x, 2 + x, 4 + x, 6 + x\}$ of nonzero elements of the maximal ideal, and the zero divisor graph is shown in Figure 2.3.

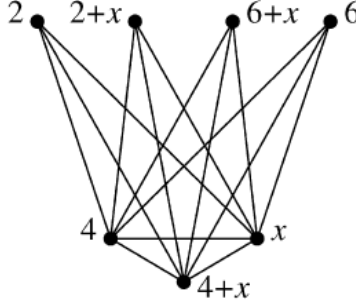


Figure 2.3: The zero-divisor graph of $\mathbb{Z}_8[X]/(2X, X^2)$.

Of course, $\Gamma(R)$ may be infinite (i.e., a ring may have an infinite number of zero-divisors). But probably $\Gamma(R)$ is of most interest when it is finite, as in this case one can draw $\Gamma(R)$. Recall that if R is finite, then each element of R is either a unit or a zero-divisor, each prime ideal of R is an annihilator ideal, and each nonunit of R is nilpotent if and only if R is local. Moreover, if R is a finite local ring with maximal ideal M , then $\text{char}(R) = p^n$ for some prime p and integer $n \geq 1$. Hence $M (= Z(R))$ is a p -group, so $|\Gamma(R)| = p^m - 1$ for some integer $m \geq 0$ (see [13, 22, 30]). We first show that $Z(R)$ is finite if and only if either R is finite or an integral domain. Also, it is shown that $\text{diam}(\Gamma(R)) \leq 3$ and $\text{gr}(\Gamma(R)) \leq 4$ or infinity. Finally, we characterize rings whose zero-divisor graphs are complete graphs.

Theorem 2.1.3 *Let R be a commutative ring. Then $\Gamma(R)$ is finite if and only if either R is finite or an integral domain. In particular, if $1 \leq |\Gamma(R)| < \infty$, then R is finite and not a field.*

Proof. Suppose that $|\Gamma(R)|$ ($=|Z(R)^*|$) is finite and nonzero. Then there are nonzero elements $x, y \in R$ with $xy = 0$. Let $I = \text{Ann}(x)$. Then $I \subset Z(R)$ is finite and $ry \in I$ for all $r \in R$. If R is infinite, then there is an $i \in I$ with $J = \{r \in R \mid ry = i\}$ infinite. For any $r, s \in J$, $(r - s)y = 0$, so $\text{Ann}(y) \subset Z(R)$ is infinite, yielding a contradiction. Thus R must be finite.

We next show that $\Gamma(R)$ is connected, $0 \leq \text{diam}(\Gamma(R)) \leq 3$ and $\text{gr}(\Gamma(R)) = 3, 4$ or ∞ .

Theorem 2.1.4 *Let R be a commutative ring. Then $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R)) \leq 3$.*

Proof. Let $x, y \in Z(R)^*$ be distinct. We have the following two cases:

Case 1: $xy = 0$. Then $d(x, y) = 1$.

Case 2: $xy \neq 0$. Then we have the following subcases:

Subcase 1: $x^2 = y^2 = 0$. If $xy = x$, then $xy = (xy)y = xy^2 = 0$, yielding a contradiction. Thus $xy \neq x$. Similarly, $xy \neq y$. Therefore, $x - xy - y$ is a path of length 2, and so $d(x, y) = 2$.

Subcase 2: $x^2 = 0$ and $y^2 \neq 0$. Then there is a $b \in Z(R)^* \setminus \{x, y\}$ with $by = 0$. If $bx = 0$, then $x - b - y$ is a path of length 2. If $bx \neq 0$, then $x - bx - y$ is a path of length 2. In either case, $d(x, y) = 2$.

Subcase 3: $y^2 = 0$ and $x^2 \neq 0$. The proof is similar to subcase 2.

Subcase 4: $x^2 \neq 0$ and $y^2 \neq 0$. Then there exist $a, b \in Z(R)^* \setminus \{x, y\}$ with $ax = by = 0$. If $a = b$, then $x - a - y$ is a path of length 2. Thus we

may assume that $a \neq b$. If $ab = 0$, then $x - a - b - y$ is a path of length 3, and hence $d(x, y) \leq 3$. If $ab \neq 0$, then $x - ab - y$ is a path of length 2 so $d(x, y) = 2$. Since in all the cases $d(x, y) \leq 3$, $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R)) \leq 3$.

Theorem 2.1.5 *Let R be a commutative ring, not necessarily with identity. If $\Gamma(R)$ contains a cycle, then $\text{gr}(\Gamma(R)) \leq 4$.*

Proof. If $\Gamma(R)$ contains a cycle $x_0 - x_1 - \cdots - x_n - x_0$ with $n \geq 4$ and $x_i x_j = 0$ for some i and j , where $j > i+1$ and either $0 \leq i < j \leq n-1$ or $1 \leq i < j \leq n$, we can eliminate x_k , where $i < k < j$, to obtain a shorter cycle. So, we may assume $\Gamma(R)$ contains a cycle $x_0 - x_1 - \cdots - x_n - x_0$ with $n \geq 4$ and $x_i x_j \neq 0$ for any i and j under the above conditions. If $x_1 x_{n-1}$ is different from x_0 and x_n , then we can form a cycle $x_0 - x_1 x_{n-1} - x_n - x_0$ of length 3. So, we may assume that $x_1 x_{n-1} = x_0$ or $x_1 x_{n-1} = x_n$. If $x_1 x_{n-1} = x_0$, then $x_0 x_{n-2} = x_1 x_{n-1} x_{n-2} = 0$, a contradiction. If $x_1 x_{n-1} = x_n$, then $x_n x_{n-2} = x_n x_1 x_{n-1} = 0$, a contradiction. Therefore, $x_0 - x_{n-2} - x_{n-1} - x_n - x_0$ is a cycle of length 4 and $n = 4$. Thus the result follows. \square

Theorem 2.1.6 *Let R be a commutative ring. Then $\Gamma(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*

Proof. (\Leftarrow) By definition.

(\Rightarrow) Suppose that $\Gamma(R)$ is a complete graph. Then for every pair of distinct elements x and y in $Z(R)$, $xy = 0$. Assume that there is an $x \in Z(R)$ with

$x^2 \neq 0$. We show that $x^2 = x$. If $x^2 \neq x$, then $x^3 = x^2x = 0$. Hence $x^2(x + x^2) = 0$ with $x^2 \neq 0$, so $x + x^2 \in Z(R)$. If $x + x^2 = x$, then $x^2 = 0$, a contradiction. Thus $x + x^2 \neq x$, thus $x^2 = x^2 + x^3 = x(x + x^2) = 0$ since $\Gamma(R)$ is a complete graph, again a contradiction. Hence $x^2 = x$. Since R has an idempotent, $R \cong R_1 \times R_2$. If $|R_1| > 2$, then since for every $1 \neq r_1 \in R_1$, $(r_1, 0)$ is a zero divisor and $\Gamma(R)$ is a complete graph, we conclude that $(1, 0)(r, 0) = (0, 0)$ for every $r \in R_1$, yielding a contradiction. Therefore $|R_1| = 2$. Similarly $|R_2| = 2$, and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

2.2 Zero-Divisor Graphs of Group Rings

Let R be a commutative ring with $1 \neq 0$ and $G \neq 1$ be a finite abelian group. Then the group ring RG is a commutative ring. Since $G \neq 1$ is finite, $\Gamma(RG) \neq \emptyset$. In this section, we characterize all finite group rings RG for which either $\text{diam}(\Gamma(RG)) \leq 2$ or $\text{gr}(\Gamma(RG)) \geq 4$ and we also investigate the genus of $\Gamma(RG)$. We begin with a useful lemma.

Lemma 2.2.1 *Let p and q be distinct primes. The cyclotomic polynomial $\phi_q(x)$ of order q is irreducible over $K = \mathbb{F}_{p^r}$ if and only if p is a generator for $(\mathbb{Z}/q\mathbb{Z})^*$ and $\gcd(r, q - 1) = 1$.*

Proof. Note that $\phi_q(x)$ factors into $\varphi(q)/d$ distinct irreducible polynomials in $K[\zeta]$, where ζ is a primitive q th root of unity and $[K[\zeta] : K] = d$. Since $[K[\zeta] : \mathbb{F}_p] = rd$, $K[\zeta] \cong \mathbb{F}_{(p^r)^d}$. Note that $\mathbb{F}_{(p^r)^d}$ is the splitting field of

polynomial $x^t - x$ where $t = p^{rd}$. We conclude that d is the least integer such that $\zeta^{(p^r)^d} = \zeta$, and thus $\zeta^{(p^r)^d - 1} = 1$. Therefore $(p^r)^d \equiv 1 \pmod{\text{ord}(\zeta)}$. Since $\text{ord}(\zeta) = q$, we have $(p^r)^d \equiv 1 \pmod{q}$. Thus $\phi_q(x)$ is irreducible if and only if $\varphi(q)/d = 1$, and so $q - 1 = d$. Hence the order of p^r in $(\mathbb{Z}/q\mathbb{Z})^*$ is $q - 1$, which is equivalent to saying that p^r is a generator of $(\mathbb{Z}/q\mathbb{Z})^*$. So p is a generator as well. Thus $\text{ord}(p^r) = q - 1 = \text{ord}(p)$, implying $\text{ord}(p)/\gcd(r, q - 1) = q - 1 = \text{ord}(p)$, so $\gcd(r, q - 1) = 1$. Finally, we conclude that $\phi_q(x)$ is irreducible if and only if p is a generator of $(\mathbb{Z}/q\mathbb{Z})^*$ and $\gcd(r, q - 1) = 1$. \square

We now characterize group rings RG with $\text{gr}(\Gamma(RG)) \geq 4$.

Proposition 2.2.2 *Let R be a finite commutative ring. Then either $\text{gr}(\Gamma(RG)) = 3$ or one of the following holds:*

- (1) $RG \cong \mathbb{Z}_2 C_2$
- (2) $RG \cong \mathbb{F}_{p^r} C_q$ where p and q are distinct primes, p is a generator for $(\mathbb{Z}/q\mathbb{Z})^*$ and $\gcd(r, q - 1) = 1$.

Proof. We divide our proof into the following cases.

Case 1: $|Z(R)| \geq 3$. Let $1 \neq g \in G$. Since $\Gamma(R)$ is connected and $|Z(R)| \geq 3$, we conclude that there exist two distinct nonzero zero-divisors $a, b \in R$ such that $ab = 0$. Then $\hat{G} - a(1 - g) - b(1 - g) - \hat{G}$ is a cycle in $\Gamma(RG)$, so $\text{gr}(\Gamma(RG)) = 3$.

Case 2: $|Z(R)| = 2$. Let $0 \neq a \in Z(R)$. Then $a^2 = 0$. Since $Ra \cong R/\text{Ann}(a)$, $|R| \leq |Ra||\text{Ann}(a)| \leq |Z(R)|^2$. Thus $|R| \leq 4$, and since R is

not a field, $|R| = 4$. If $|G| > 2$ then there exist nontrivial distinct elements $g, h \in G$. Hence $\hat{G} - a(1 - g) - a(1 - h) - \hat{G}$ is a cycle in $\Gamma(RG)$, so $gr(\Gamma(RG)) = 3$. Next assume that $|G| = |\langle g \rangle| = 2$. If $Char(R) = 2$ then $\hat{G}^2 = 0$. Let $0 \neq a, b \in R$ be such that $1 \neq a \neq b \neq 1$. Then $\hat{G} - a\hat{G} - b\hat{G} - \hat{G}$ is a cycle in $\Gamma(RG)$, so $gr(\Gamma(RG)) = 3$. If $Char(R) = 4$ then $\hat{G} - (1 - g) - 2\hat{G} - \hat{G}$ is a cycle in $\Gamma(RG)$, so $gr(\Gamma(RG)) = 3$ again.

Case 3: $|Z(R)| = 1$, i.e., R is an integral domain. Since R is finite, R is a field. We divide the proof into four subcases.

Subcase 1: $Char(R) || G|$ and $|R| > 3$. Then $\hat{G}^2 = |G|\hat{G} = 0$. Let $r, s \in R$ be two nonzero elements such that $1 \neq r \neq s \neq 1$. Then $\hat{G} - r\hat{G} - s\hat{G} - \hat{G}$ is a cycle in $\Gamma(RG)$, so $gr(\Gamma(RG)) = 3$.

Subcase 2: $Char(R) || G|$ and $|R| = 3$. Then $Char(R) = 3$, and so $|G| \geq 3$. Let $g \in G$ be a nontrivial element. Then $\hat{G} - (1 - g) - 2\hat{G} - \hat{G}$ is a cycle in $\Gamma(RG)$, so $gr(\Gamma(RG)) = 3$.

Subcase 3: $Char(R) || G|$ and $|R| = 2$. Suppose that $|G| > 2$. Since $Char(R) = 2$ and $Char(R) || G|$, we conclude that there exists a subgroup H of G of order 2. Let $g \in G \setminus H$. Then $\hat{G} - g\hat{H} - \hat{H} - \hat{G}$ is a cycle in $\Gamma(RG)$, so $gr(\Gamma(RG)) = 3$. If $|G| = 2$ then $R \cong \mathbb{Z}_2$ and $G \cong C_2$, so $gr(\Gamma(RG)) = \infty$.

Subcase 4: $Char(R) \nmid |G| = n$. Therefore, by the Perlis-Walker Theorem [29, Theorem 3.5.4], $RG \cong \bigoplus_{d|n} a_d R(\zeta_d)$ where ζ_d is a primitive root of unity of order d and $a_d = \frac{n_d}{[R(\zeta_d):R]}$, with n_d denoting the number of elements of order d in G . Thus, if n is not a prime then RG is a direct product of at least three fields, so $gr(\Gamma(RG)) = 3$. Next assume that $|G| = q$ and $Char(R) = p$

where q and p are primes. Thus $p \neq q$. Note that $RG \cong R[x]/(x^q - 1)$. If the cyclotomic polynomial $\phi_q(x)$ of order q is reducible, then since $x^q - 1 = (x - 1)\phi_q(x)$, we conclude that RG is a direct product of at least three fields, so $gr(\Gamma(RG)) = 3$. If $\phi_q(x)$ is irreducible over R , then by Lemma 2.2.1 $R \cong \mathbb{F}_{p^r}$, where p is a generator for $(\mathbb{Z}/q\mathbb{Z})^*$ and $\gcd(r, q - 1) = 1$. In this subcase, $gr(\Gamma(RG)) = 4$ or $gr(\Gamma(RG)) = \infty$. \square

By Lemma 2.1.4, $\Gamma(RG)$ is connected and $diam(RG) \leq 3$. If $diam(\Gamma(RG)) \leq 1$, then $\Gamma(RG)$ is a complete graph. In Proposition 2.2.3, we characterize group rings with complete zero-divisor graphs. More generally, in Theorem 2.2.4 we characterize group rings RG with $diam(\Gamma(RG)) \leq 2$.

Proposition 2.2.3 *$\Gamma(RG)$ is a complete graph if and only if R is an integral domain with $Char(R) = 2$ and $G \cong C_2$.*

Proof. Let $\Gamma(RG)$ be a complete graph. Assume to the contrary that R is not an integral domain. Then there exist nonzero elements $a, b \in R$ such that $ab = 0$. Since \hat{G} is a zero-divisor of RG and $diam(\Gamma(RG)) \leq 1$, we conclude that $a\hat{G} = \sum_{g \in G} ag = 0$, hence $a = 0$, yielding a contradiction. Therefore R is an integral domain.

It is easy to see that $RG \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus it follows from Theorem 2.1.6 that $(Z(RG))^2 = 0$. Let $1 \neq g \in G$. Since $(1 - g)\hat{G} = 0$, $(1 - g) \in Z(RG)$. Hence $(1 - g)(1 - g) = 1 - 2g + g^2 = 0$, so $g^2 = 1$ (for otherwise, if $g^2 \neq 1$, then the coefficient of the identity element 1 is not zero, thus $1 - 2g + g^2 \neq 0$, yielding a contradiction). Now we have $2 - 2g = 0$, which implies $2 = 0$.

Thus $\text{Char}(R) = 2$. Next, suppose that $|G| > 2$ and g_1, g_2 are two distinct nontrivial elements of G . Since $(1 - g_1)\hat{G} = (1 - g_2)\hat{G} = 0$, both $(1 - g_1)$ and $(1 - g_2)$ are zero-divisors. Thus $(1 - g_1)(1 - g_2) = 1 - g_1 - g_2 + g_1g_2 = 0$. As above we have $g_1 = g_2$, yielding a contradiction. Therefore $|G| = 2$.

Conversely, assume that R is an integral domain with $\text{Char}(R) = 2$ and $|G| = 2$. Let $a + bg$ be a nonzero zero-divisor of RG such that $a \neq b$. Then there exists a nonzero element $c + dg \in RG$ such that $(a + bg)(c + dg) = 0$. Hence $ac + bd = 0$ and $ad + bc = 0$, so $ac + bd + ad + bc = 0$. Therefore $a(c + d) + b(c + d) = 0$, so $(a + b)(c + d) = 0$. Since R is an integral domain with $\text{Char}(R) = 2$ and $a \neq b$, we have $c = d$. Since $(a + bg)(c + cg) = 0$, $ac + cb = 0$, so $c(a + b) = 0$. Since $c \neq 0$ and R is an integral domain, we conclude that $a + b = 0$, so $a = b$, yielding a contradiction. Therefore, if $a + bg \in Z(RG)$ then $a = b$. Since $(1 + g)(1 + g) = 0$, we obtain that for every pair $x, y \in R$, $(x + xg)(y + yg) = xy(1 + g)(1 + g) = 0$, so $\Gamma(RG)$ is a complete graph. \square

Theorem 2.2.4 *Let R be a finite commutative ring and G be a finite abelian group. Then $\text{diam}(\Gamma(RG)) \leq 2$ if and only if either R is a local ring and G is a p -group such that $p \in J(R)$ or $RG \cong \mathbb{F}_{p^r}C_q$ where p and q are distinct primes, p is a generator for $(\mathbb{Z}/q\mathbb{Z})^*$ and $\gcd(r, q - 1) = 1$.*

Proof. If $\text{diam}(\Gamma(RG)) \leq 1$, then by Proposition 2.2.3, R is an integral domain with $\text{Char}(R) = 2$ and $G \cong C_2$. Since R is finite, R is a field with $\text{Char}(R) = 2$. So R is a local ring and $2 \in J(R)$. Assume that

$\text{diam}(\Gamma(RG)) = 2$. Since RG is an Artinian ring, by Theorem 1.3.4 $RG \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i ($1 \leq i \leq n$) is a local ring. If $n \geq 3$ then $d((1, 1, \dots, 1, 0), (0, 1, \dots, 1)) = 3$, yielding a contradiction. Therefore $n \leq 2$. If $n = 1$ then RG is a local ring, so by [32, Theorem] R is a local ring, G is a p -group and $p \in J(R)$. If $n = 2$ then $Z(RG)$ is not an ideal. It follows from [11, Theorem 2.7 and Lemma 2.1] that $gr(\Gamma(RG)) \geq 4$. Therefore, by Proposition 2.2.2, $RG \cong \mathbb{F}_{p^r}C_q$, where p and q are distinct primes, p is a generator for $(\mathbb{Z}/q\mathbb{Z})^*$ and $\gcd(r, q - 1) = 1$.

Conversely, if R is a local ring and G is a p -group such that $p \in J(R)$ then by [32, Theorem], RG is a local ring. Note that the annihilator of the unique maximal ideal $J(R) + \Delta(G)$ of RG is $\text{Ann}_R(J(R))\hat{G}$ (where $\text{Ann}_R(J(R))$ is the annihilator of $J(R)$ in R). Thus \hat{G} is adjacent to every vertex of $\Gamma(RG)$, so $\text{diam}(\Gamma(RG)) \leq 2$. If $RG \cong \mathbb{F}_{p^r}C_q$ where p and q are distinct primes, p is a generator for $(\mathbb{Z}/q\mathbb{Z})^*$ and $\gcd(r, q - 1) = 1$, then RG is a direct product of two fields, so $\text{diam}(\Gamma(RG)) = 2$. \square

In [7], the authors generalized Wickham's theorem to arbitrary (not necessary finite) commutative rings. The following proposition is a special case of [7, Corollary 3.7] regarding group rings and we shall provide a short proof for it.

Proposition 2.2.5 *For each positive integer r , there are finitely many commutative group rings RG over finite groups G with $\gamma(\Gamma(RG)) = r$.*

Proof. Let $\gamma(\Gamma(RG)) = r$. It is sufficient to show that $|RG|$ is bounded above by a constant depending only on r . Assume that $|R| > 3$. Then $|R\hat{G}| > 3$. Let a, b, c be non-zero elements of $R\hat{G}$. Since $a\Delta(G) = b\Delta(G) = c\Delta(G) = 0$, we conclude that $K_{|\Delta(G)|-4,3}$ is a subgraph of $\Gamma(RG)$. So by the formula for genus of complete bipartite graphs, $\lceil |\Delta(G)| - 6 \rceil / 4 \leq r$. Hence $|\Delta(G)| \leq 4r + 6$. Since $R \cong RG/\Delta(G)$, $|RG| \leq |R||\Delta(G)| \leq |\Delta(G)|^2$. Thus $|RG| \leq (4r + 6)^2$ and we are done.

Next, we may assume that $|R| = p \leq 3$.

Case 1: $p \mid |G|$. We may assume that $|G| \geq 2^3$. Let H be a subgroup of G of order $|G|/p$. Note that by Lemma 1.3.8 $\hat{H}\Delta(G, H) = \hat{G}\Delta(G, H) = (\hat{G} + \hat{H})\Delta(G, H) = 0$. We conclude that $K_{|\Delta(G, H)|-4,3}$ is a subgraph of $\Gamma(RG)$. So by the formula for genus of complete bipartite graphs, $\lceil |\Delta(G, H)| - 6 \rceil / 4 \leq r$. Hence $|\Delta(G, H)| \leq 4r + 6$. Since $|RG| = |R(G/H)||\Delta(G, H)| = |R|^p |\Delta(G, H)| \leq |\Delta(G, H)|^{p+1}$ (where $|G/H| = p$ and $|R| \leq |\Delta(G, H)|$), we conclude that $|RG| \leq (4r + 6)^4$ and we are done.

Case 2: $p \nmid |G|$. Then by [29, Theorem 3.5.4] $RG \cong R \times F_1 \times \cdots \times F_n$, where each F_i is a field. Without loss of generality, we may assume that $|F_i| \leq |F_{i+1}|$. If $n = 1$, then since $|R| \leq 3$, $\Gamma(RG) \cong \Gamma(R \times F_1) \cong K_{|R|-1, |F_1|-1}$, and thus $\gamma(\Gamma(RG)) = \gamma(K_{|R|-1, |F_1|-1}) = 0$. If $n \geq 2$, then since $(R \times F_1 \times 0 \cdots \times 0)(0 \times 0 \times F_2 \times \cdots \times F_n) = (0)$, we conclude that $K_{3, |F_2 \times \cdots \times F_n|}$ is a subgraph of $\Gamma(RG)$. Therefore, as before we have $\lceil |F_2 \times \cdots \times F_n| - 6 \rceil / 4 \leq r$, and thus $|F_2 \times \cdots \times F_n| \leq 4r + 6$. Hence $|RG| = |R||F_1||F_2 \times \cdots \times F_n| \leq p|F_2 \times \cdots \times F_n|^2 \leq p(4r + 6)^2$ as desired. \square

2.3 The Isomorphism Problem for Zero-divisor Graphs of Group Rings

For a not necessarily commutative ring R , we define a simple undirected graph $\bar{\Gamma}(R)$ with vertex set $D(R)$ (the set of all non-zero zero-divisors of R) in which two distinct vertices x and y are adjacent if and only if either $xy = 0$ or $yx = 0$ (see [34]). In this section, we investigate the isomorphism problem for zero-divisor graphs of group rings, that is, the problem of when $\bar{\Gamma}(R_1G) \cong \bar{\Gamma}(R_2H)$ implies that $R_1G \cong R_2H$ (or possibly, $R_1 \cong R_2$ and $G \cong H$). We show that the rank and the cardinality of a finite abelian p -group are determined by the zero-divisor graph of its modular group ring. It is also shown that the isomorphism problem for zero-divisor graphs of finite noncommutative reversible group rings and semisimple group rings has an affirmative answer. We first recall some known results.

Lemma 2.3.1 [5, Theorems 14, 16, and 17] *Let K, K_1 be two finite fields and G, G_1 be two finite groups such that $\bar{\Gamma}(KG) \cong \bar{\Gamma}(K_1G_1)$. Then the following hold:*

- (1) $K \cong K_1$ and $|G| = |G_1|$.
- (2) If G is an abelian group, then so is G_1 .
- (3) If G is a cyclic group, then $G \cong G_1$.

Lemma 2.3.2 *Let K be a field of characteristic p and G be a finite p -group. Then KG is a Gorenstein local ring.*

Proof. By [32, Theorem], KG is a local ring. Since $K \cong KG/\Delta(G)$, we conclude that $\Delta(G)$ is the unique maximal ideal of KG . Since $\text{Ann}(\Delta(G)) = K\hat{G}$, $\dim_K(\text{Ann}(\Delta(G))) = 1$, so KG is a Gorenstein local ring. \square

The following proposition is a special case of Lemma 2.3.1 (1) and we shall provide a short proof for it.

Proposition 2.3.3 *Let K, K_1 be two finite fields of characteristic p and G, G_1 be two finite p -groups. If $\bar{\Gamma}(KG) \cong \bar{\Gamma}(K_1G_1)$, then $K \cong K_1$ and $|G| = |G_1|$.*

Proof. Since $|\{v \in V(\bar{\Gamma}(KG)) | \deg(v) = m(\bar{\Gamma}(KG))\}|$ is equal to $|\text{Ann}(\Delta(G))| - 1 = |K| - 1$, and $|\{w \in V(\bar{\Gamma}(K_1G_1)) | \deg(w) = m(\bar{\Gamma}(K_1G_1))\}|$ is equal to $|\text{Ann}(\Delta(G_1))| - 1 = |K_1| - 1$, we obtain that $|K| = |K_1|$, so $K \cong K_1$. Since $|K|^{|G|-1} = |\Delta(G)| = |\Delta(G_1)| = |K_1|^{|G_1|-1}$. We have $|\Delta(G)| = |\Delta(G_1)|$, so $|G| = |G_1|$. \square

Let R be a commutative ring. As in [31, 3.5], for $x, y \in R$, we define that x is equivalent to y , denoted by $x \sim y$, if $\text{Ann}(x) = \text{Ann}(y)$. Clearly, \sim is an equivalence relation on R , and its restriction to $\Gamma(R)$ is also an equivalence relation.

Let K be a field of characteristic p . Let G be a finite group and $H = H_p \times H_1$ be an abelian group, where H_p is the Sylow p -subgroup of H . If $\bar{\Gamma}(KG) \cong \bar{\Gamma}(KH)$, then we can prove that $G = G_p \times G_1$ such that $\bar{\Gamma}(KG_p) = \bar{\Gamma}(KH_p)$ and $K_1G_1 \cong K_1H_1$ (see Proposition 2.3.5 for details). Thus, the question of determining when $\bar{\Gamma}(KG) \cong \bar{\Gamma}(KH)$ implies $KG \cong KH$ is essentially

reduced to that of determining when $\bar{\Gamma}(K_i G_p) \cong \bar{\Gamma}(K_i H_p)$ implies $K_i G_p \cong K_i H_p$ for $1 \leq i \leq m$, where $KH_1 \cong K_1 \times K_2 \times \cdots \times K_m$ as in the proof of Proposition 2.3.5. Here is our first main result in this section.

Theorem 2.3.4 *Let K be a field of order p , K_1 be a finite field of characteristic p , G and H be two finite abelian p -groups. If $\Gamma(KG) \cong \Gamma(K_1 H)$, then $K \cong K_1$, $|G| = |H|$, and $\text{rank}(G) = \text{rank}(H)$.*

Proof. By Proposition 2.3.3, $K \cong K_1$ and $|G| = |H|$. So we may assume that $K = K_1$ in the rest of proof.

Let $\text{rank}(G) = n$, $\text{rank}(H) = m$ and $G = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$. It follows from [29, Lemma 3.3.2] that $\Delta(G)$ is generated by $\{g_1 - 1, g_2 - 1, \dots, g_n - 1\}$ as a KG -module. We also note that $\Delta(G)$ is a nilpotent ideal.

By Lemma 2.3.2, KG is a local ring. Since $K \cong KG/\Delta(G)$, we conclude that $\Delta(G)$ is the unique maximal ideal of KG . Let I be an ideal of KG such that $|I| = p^{|G|-2}$. We note that the image of the set $\{g_1 - 1, g_2 - 1, \dots, g_n - 1\}$ in $\Delta(G)/\Delta(G)^2$ forms a basis for $\Delta(G)/\Delta(G)^2$ over K . Thus $\Delta(G)/\Delta(G)^2$ is a vector space of dimension n over K . If $\Delta(G)^2 \not\subseteq I$, then $I + \Delta(G)^2 = \Delta(G)$. By Nakayama's Lemma $I = \Delta(G)$, yielding a contradiction since $|\Delta(G)| = p^{|G|-1} > p^{|G|-2} = |I|$. Thus $\Delta(G)^2 \subseteq I$ and $\dim_K(I/\Delta(G)^2) = n - 1$. Let $\chi : \{I \trianglelefteq KG : |I| = p^{|G|-2}\} \rightarrow \{\text{maximal subspaces of } \Delta(G)/\Delta(G)^2\}$ be a function defined by $\chi(I) = I/\Delta(G)^2$. We shall prove that χ is a bijection. Thus the number of maximal subspaces of $\Delta(G)/\Delta(G)^2$ is equal to the number of ideals of KG of order $p^{|G|-2}$. Let V_1 be a maximal subspace

of $\Delta(G)/\Delta(G)^2$, $\{x_1 + \Delta(G)^2, x_2 + \Delta(G)^2, \dots, x_{n-1} + \Delta(G)^2\}$ a basis for V_1 , and let $I_1 = KGx_1 + KGx_2 + \dots + KGx_{n-1} + \Delta(G)^2$. Then $V_1 = I_1/\Delta(G)^2$, where I_1 is an ideal of order $p^{|G|-2}$ and so χ is surjective. Let $\chi(I) = I/\Delta(G)^2 = J/\Delta(G)^2 = \chi(J)$ where I and J are ideals of order $p^{|G|-2}$. Since $\Delta(G)^2 \subseteq I \cap J$, we conclude that $I = J$. Thus χ is injective, so it is a bijection. Since the number of maximal subspaces is equal to $1+p+\dots+p^{n-1}$, we conclude that KG has $1+p+\dots+p^{n-1}$ ideals of order $p^{|G|-2}$.

Let G_1 and G_2 be maximal subgroups of G . Then $\dim_K((\Delta(G, G_1) + \Delta(G)^2)/\Delta(G)^2) = n-1$, so $|\Delta(G, G_1) + \Delta(G)^2| = p^{|G|-2}$. Similarly, we have $|\Delta(G, G_2) + \Delta(G)^2| = p^{|G|-2}$. Since $\Delta(G, G_1) + \Delta(G, G_2) + \Delta(G)^2 = \Delta(G)$ we obtain that $\Delta(G, G_1) + \Delta(G)^2$ and $\Delta(G, G_2) + \Delta(G)^2$ are two different ideals of order $p^{|G|-2}$. Since G has $1+p+\dots+p^{n-1}$ maximal subgroups, we conclude that $\{\Delta(G, G_1) + \Delta(G)^2 : G_1 \text{ is a maximal subgroup of } G\}$ is the set of all the ideals of KG of order $p^{|G|-2}$.

Let $|\langle g_i \rangle| = p^{e_i}$ for each i . Let G_1 be a maximal subgroup of G with invariants $p^{e_1'}, \dots, p^{e_n'}$. Therefore $e_i' = e_i$ for all i except for one l , i.e., $e_i' = e_i$ for all $i \neq l$ and $e_l' = e_l - 1$. Without loss of generality, we may assume that $l = 1$. Thus $e_1' = e_1 - 1$ and $e_i' = e_i$ for all $i \geq 2$. Hence, $G_1 = \langle g_1^p \rangle \langle g_2 \rangle \dots \langle g_n \rangle$. Let $x = (g_1 - 1)^{p^{e_1}-2} (g_2 - 1)^{p^{e_2}-1} \dots (g_n - 1)^{p^{e_n}-1}$. Then since $x(g_1 - 1) \neq 0$, we have $x \notin K\hat{G}$. By [33, Theorem 1], $\Delta(G)^{1+\sum_{i=1}^n (p^{e_i}-1)} = 0$. Since $x\Delta(G)^2 \subset \Delta(G)^{1+\sum_{i=1}^n (p^{e_i}-1)} = 0$ and $x\Delta(G, G_1) = 0$, we conclude that $x \in \text{Ann}(\Delta(G, G_1) + \Delta(G)^2) \setminus K\hat{G}$. Since $\Delta(G, G_1) + \Delta(G)^2 \subseteq \text{Ann}(x) \subseteq \Delta(G)$, we obtain that either $\text{Ann}(x) = \Delta(G)$ or $\text{Ann}(x) = \Delta(G, G_1) +$

$\Delta(G)^2$. If $\text{Ann}(x) = \Delta(G)$, then $x \in K\hat{G}$, yielding a contradiction. Therefore $\Delta(G, G_1) + \Delta(G)^2 = \text{Ann}(x)$, and thus every ideal of order $p^{|G|-2}$ is an annihilator of a zero-divisor of KG .

Let v be a vertex of degree $p^{|G|-2} - 1$ or $p^{|G|-2} - 2$ of $\Gamma(KG)$. Then $|\text{Ann}(v)|$ is equal to $p^{|G|-2}$. Let $V = \{v_1, v_2, \dots, v_t\}$ be the set of all the vertices of $\Gamma(KG)$ of degree $p^{|G|-2} - 1$ or $p^{|G|-2} - 2$, and $N_{\Gamma(KG)}(v_i) = \{ \text{all of vertices of } \Gamma(KG) \text{ which are adjacent to } v_i \}$ for each i . Recall that there is an equivalence relation \sim on V such that for $v_1, v_2 \in V$, $v_1 \sim v_2$ if and only if $\text{Ann}(v_1) = \text{Ann}(v_2)$. Let $[v_i]$ denote the equivalence class of v_i in V and let $\bar{V} = \{[v_1], [v_2], \dots, [v_r]\}$ be the set of all the equivalence classes in V . We now show that any isomorphism $f : \Gamma(KG) \rightarrow \Gamma(KH)$ preserves this equivalence relation.

Let $x, y \in V$ be such that $x \sim y$. Assume that $f(x) = x_1$ and $f(y) = y_1$. We shall show that $x_1 \sim y_1$. Since $\text{Ann}(x) = \text{Ann}(y)$, $N_{\Gamma(KG)}(x) \setminus \{y\} = N_{\Gamma(KG)}(y) \setminus \{x\}$. Thus $N_{\Gamma(KH)}(x_1) \setminus \{y_1\} = N_{\Gamma(KH)}(y_1) \setminus \{x_1\}$. First assume $xy = 0$, so $x_1y_1 = 0$. Note that either $x_1 + \hat{H} = y_1$ or $x_1 + \hat{H} \in N_{\Gamma(KH)}(y_1) \setminus \{x_1\} = N_{\Gamma(KH)}(x_1) \setminus \{y_1\}$. In both cases, we obtain that $(x_1)^2 = x_1(x_1 + \hat{H}) = 0$. By a symmetric argument, we obtain $(y_1)^2 = 0$. Thus $\text{Ann}(x_1) = \{N_{\Gamma(KH)}(x_1) \setminus \{y_1\}, y_1, x_1, 0\} = \{N_{\Gamma(KH)}(y_1) \setminus \{x_1\}, x_1, y_1, 0\} = \text{Ann}(y_1)$. Therefore $x_1 \sim y_1$ as desired. Next we assume that $xy \neq 0$. If $(x_1)^2 = 0$, then either $x_1 + \hat{H} = y_1$ or $x_1 + \hat{H} \in N_{\Gamma(KH)}(x_1) \setminus \{y_1\} = N_{\Gamma(KH)}(y_1) \setminus \{x_1\}$. In the first case we have $x_1y_1 = x_1(x_1 + \hat{H}) = (x_1)^2 = 0$, yielding a contradiction since $xy \neq 0$. In the latter case we have $x_1y_1 = (x_1 + \hat{H})y_1 = 0$, yielding

a contradiction again. Thus we must have $(x_1)^2 \neq 0$. By a symmetric argument, we have $(y_1)^2 \neq 0$. Therefore $\text{Ann}(x_1) = \{N_{\Gamma(KH)}(x_1) \setminus \{y_1\}, 0\} = \{N_{\Gamma(KH)}(y_1) \setminus \{x_1\}, 0\} = \text{Ann}(y_1)$. Again, $x_1 \sim y_1$ as desired. We have shown that any isomorphism $f : \Gamma(KG) \rightarrow \Gamma(KH)$ preserves this equivalence relation.

Since every ideal of order $p^{|G|-2}$ is an annihilator of a zero-divisor v_i of KG , we conclude that $|\overline{V}|$ is equal to the number of ideals of order $p^{|G|-2}$. So $|\overline{V}|$ is equal to the number of maximal subgroups of G . Let $W = \{w_1, w_2, \dots, w_s\}$ be the set of all the vertices of $\Gamma(KH)$ of degree $p^{|G|-2} - 1$ or $p^{|G|-2} - 2$, and $\overline{W} = \{[w_1], [w_2], \dots, [w_q]\}$ be the set of all of equivalence classes in W . Then $|\overline{W}|$ is equal to the number of maximal subgroups of H . Since the isomorphism $f : \Gamma(KG) \rightarrow \Gamma(KH)$ preserves this equivalence relation, it induces a bijection between \overline{V} and \overline{W} , so $|\overline{V}| = |\overline{W}|$. Therefore, the number of maximal subgroups of G is equal to the number of maximal subgroups of H . Thus $1 + p + \dots + p^{n-1} = 1 + p + \dots + p^{m-1}$. So $\text{rank}(G) = n = m = \text{rank}(H)$. \square

Proposition 2.3.5 *Let K and K_1 be two fields of characteristic p . Let G be a finite group and $H = H_p \times H_1$ be a finite abelian group, where H_p is the sylow p -subgroup of H . If $\overline{\Gamma}(KG) \cong \overline{\Gamma}(K_1H)$, then $K \cong K_1$ and $G = G_p \times G_1$ such that $\overline{\Gamma}(K_1G_p) \cong \overline{\Gamma}(K_1H_p)$ and $G_1 \cong H_1$.*

Proof. It follows from Lemma 2.3.1 that $|G| = |H|$ and $K \cong K_1$. So we may assume that $K = K_1$ in the rest of the proof. Since $|G| = |H|$

and G is abelian (by Lemma 2.3.1), we conclude that $G = G_p \times G_1$, where $|G_p| = |H_p|$ and $|G_1| = |H_1|$. Since KH_1 is a semisimple group ring, $KH_1 \cong K_1 \times K_2 \times \cdots \times K_m$, where K_i 's are fields and $K_1 \cong K$. So $K(H_p \times H_1) \cong (KH_1)H_p \cong (K_1 \times K_2 \times \cdots \times K_m)H_p \cong K_1H_p \times K_2H_p \times \cdots \times K_mH_p$. Similarly $K(G_p \times G_1) \cong (KG_1)G_p \cong (S_1 \times S_2 \times \cdots \times S_n)G_p \cong S_1G_p \times S_2G_p \times \cdots \times S_nG_p$, where S_i 's are fields and $S_1 \cong K$. Since $\bar{\Gamma}(KH) \cong \bar{\Gamma}(KG)$, $\bar{\Gamma}(K_1H_p \times K_2H_p \times \cdots \times K_mH_p) \cong \bar{\Gamma}(S_1G_p \times S_2G_p \times \cdots \times S_nG_p)$. By [32, Theorem], K_iH_p and S_jG_p (for each i and j) are local, so K_iG_p and S_jH_p are indecomposable. By [5, Theorem 11] and [24, Lemma 3.8], we obtain that $m = n$ and after a permutation of indices $\bar{\Gamma}(K_iH_p) \cong \bar{\Gamma}(S_iG_p)$ for each i . Thus $\bar{\Gamma}(K_1H_p) \cong \bar{\Gamma}(K_1G_p)$. Also, by [5, Theorem 16], $K_i \cong S_i$. Therefore $K_1H_1 \cong K_1G_1$.

The following theorem shows that the isomorphism problem for zero-divisor graphs of integral group rings has a negative answer.

Theorem 2.3.6 *Let p be any prime and C_p be the cyclic group of order p . Then $\Gamma(\mathbb{Z}C_p) \cong \Gamma(\mathbb{Q}C_p) \cong K_{|\mathbb{Z}|, |\mathbb{Z}|}$, a complete bipartite graph. In particular, for distinct primes p and q , we have $\Gamma(\mathbb{Z}C_p) \cong \Gamma(\mathbb{Z}C_q)$, but $\mathbb{Z}C_p \not\cong \mathbb{Z}C_q$.*

Proof. Since any cyclotomic polynomial of prime order p over \mathbb{Q} is irreducible, $\mathbb{Q}C_p \cong K_1 \times K_2$, where K_1 and K_2 are infinite fields. Notice that $\mathbb{Z}C_p$ is a subring of a reduced ring $\mathbb{Q}C_p$. We conclude that $\mathbb{Z}C_p$ is a reduced ring. Since $\Gamma(\mathbb{Z}C_p)$ is a subgraph of $\Gamma(K_1 \times K_2)$, we obtain that $gr(\Gamma(\mathbb{Z}C_p)) = 4$ or $gr(\Gamma(\mathbb{Z}C_p)) = \infty$. Let $1 \neq a \in C_p$. Then $(1 - a) - 2\hat{C}_p - 3(1 - a) - 4(\hat{C}_p)$

is a cycle in $\Gamma(\mathbb{Z}C_p)$, and so $gr(\Gamma(\mathbb{Z}C_p)) = 4$. Since $\mathbb{Z}C_p$ is a reduced ring and $gr(\Gamma(\mathbb{Z}C_p)) = 4$, by [11, Theorem 2.2], $\Gamma(\mathbb{Z}C_p)$ is a complete bipartite graph. Since $(\mathbb{Z}\hat{C}_p)\Delta(C_p) = 0$, $K_{|\mathbb{Z}\hat{C}_p|, |\Delta(C_p)|}$ is a subgraph of $\Gamma(\mathbb{Z}C_p)$. Thus $\Gamma(\mathbb{Z}C_p) \cong K_{|\mathbb{Z}\hat{C}_p|, |\Delta(C_p)|} \cong K_{|\mathbb{Z}|, |\mathbb{Z}|} \cong \Gamma(\mathbb{Q}C_p)$. \square

Next we show that the isomorphism problem for zero-divisor graphs of noncommutative reversible group rings has an affirmative answer.

Lemma 2.3.7 *Let R be a finite ring and G be a finite non-abelian group. If RG is reversible, then $RG \cong (\Pi_{i=1}^n K_i)Q_8$, where each K_i is a field of characteristic 2.*

Proof. Since RG is reversible and G is non-abelian, it was proved in [26] that $RG = R(Q_8 \times H)$, where $G = Q_8 \times H$ is a Hamiltonian group. Note that $RG = R(Q_8 \times H) \cong (RH)Q_8$ is reversible. It follows from [27, Theorem 2.1] that $RG \cong (\Pi_{i=1}^n K_i)Q_8$, where each $K_i \cong GF(2^{n_i})$ (the Galois field of order 2^{n_i}) with n_i odd. \square

Theorem 2.3.8 *Let R, S be two finite rings, and G, H be two finite non-abelian groups such that RG and SH are reversible group rings. If $\bar{\Gamma}(RG) \cong \bar{\Gamma}(SH)$, then $RG \cong SH$.*

Proof. By Lemma 2.3.7, $RG \cong (\Pi_{i=1}^n F_i)Q_8$, where each F_i is a field of characteristic 2 and $RH \cong (\Pi_{j=1}^m E_j)Q_8$, where each E_j is a field of characteristic 2. Since for each pair of i and j F_iQ_8 and E_jQ_8 are local (by [32, Theorem]), we conclude that F_iQ_8 and E_jQ_8 are indecomposable. By [5, Theorem 11]

and [24, Lemma 3.8], we obtain that $m = n$ and after a permutation of indices, $\bar{\Gamma}(K_i Q_8) \cong \bar{\Gamma}(E_i Q_8)$ for each i . It follows from [5, Theorem 16] that $K_i \cong E_i$ for each i . Therefore $RG \cong SH$. \square

Finally, we show that two finite semisimple group rings are isomorphic if and only if their zero-divisor graphs are isomorphic.

Proposition 2.3.9 *Let RG be a (not necessarily commutative) finite group ring and S be a finite semisimple ring. If $\bar{\Gamma}(RG) \cong \bar{\Gamma}(S)$, then $RG \cong S$.*

Proof. Since G is finite, $\hat{G} \in Z(RG) \neq \{0\}$, so $\bar{\Gamma}(RG) \neq \emptyset$. Since $\bar{\Gamma}(RG) \cong \bar{\Gamma}(S)$ and $\bar{\Gamma}(RG) \neq \emptyset$, $\bar{\Gamma}(S) \neq \emptyset$. We first investigate the following cases:

Case 1: $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and RG is commutative. Since $\bar{\Gamma}(RG) \cong \bar{\Gamma}(S)$ we conclude that $\bar{\Gamma}(RG)$ is a complete graph of two vertices, so $|Z(RG)| = 3$ and $gr(RG) = \infty$. By Proposition 2.2.2, $RG \cong \mathbb{Z}_2 C_2$, yielding a contradiction since $|Z(\mathbb{Z}_2 C_2)| = 2 \neq 3$.

Case 2: $S \cong \mathbb{Z}_6$ and RG is commutative. Since there is no group ring RG with 3 nonzero zero-divisors, this case is impossible.

Case 3: $S \cong \mathbb{Z}_2 \times K$, where K is a field and RG is commutative. Then $\bar{\Gamma}(S) = \Gamma(S) = K_{1,|K|-1}$. Since $\Gamma(S)$ is a star graph and $\bar{\Gamma}(RG) = \Gamma(RG) \cong \Gamma(S)$, we obtain that $\Gamma(RG)$ is a star graph, so $gr(\Gamma(RG)) = \infty$. By the proof of Proposition 2.2.2, we conclude that $RG \cong K_1 \times K_2$, where K_1 and K_2 are two fields. If $K_1 \not\cong \mathbb{Z}_2$ and $K_2 \not\cong \mathbb{Z}_2$ then $\Gamma(RG)$ is not a star graph, yielding a contradiction. Thus, without loss of generality, we may

assume that $K_1 \cong \mathbb{Z}_2$, so $RG \cong \mathbb{Z}_2 \times K_2$, and thus $\Gamma(RG) \cong K_{1,|K_2|-1}$. Since $\Gamma(RG) \cong \Gamma(S)$ we conclude that $|K_2| = |K|$, so $K_2 \cong K$. Therefore $RG \cong S$.

Now, by [4, Theorem 21] either $RG \cong S$ or one of the above cases occurs. Since Cases 1 and 2 are impossible and in Case 3 $RG \cong S$, we must have $RG \cong S$. \square

We conclude this chapter with the following corollary.

Corollary 2.3.10 *Two finite semisimple group rings are isomorphic if and only if their zero-divisor graphs are isomorphic.*

Chapter 3

Artinian Rings Whose Annihilating-Ideal Graphs Have Positive Genus

Let R be a commutative ring and $\mathbb{A}(R)$ be the set of ideals with non-zero annihilators. The annihilating-ideal graph of R is defined as the graph $\mathbb{AG}(R)$ with vertex set $\mathbb{A}(R)^* = \mathbb{A} \setminus \{(0)\}$ such that two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Thus, $\mathbb{AG}(R)$ is the empty graph if and only if R is an integral domain. The notion of annihilating-ideal graph was first introduced and systematically studied in [15, 16]. Recently it has received a great deal of attention from several authors, for instance, [1, 2, 6]. Here we investigate commutative Artinian rings R whose annihilating-ideal graphs have positive genus $\gamma(\mathbb{AG}(R))$. Several authors recently investigated the

genus of a zero-divisor graph (for instance, see [8, 41, 43, 44]). In particular in [44, Theorem 2], it was shown that for any positive integer g , there are only finitely many finite commutative rings whose zero-divisor graphs have genus g . In this chapter it is shown that if R is a Artinian commutative ring such that $0 < \gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then R has only finitely many ideals, extending a recent result in [6].

3.1 Preliminary

All rings are assumed to be commutative in this section. We first list a few preliminary results which are needed to prove our main result. The following useful remark will be used frequently in the sequel.

Remark 3.1.1 It is well known that if V is a vector space over an infinite field \mathbb{F} , then V can not be the union of finitely many proper subspaces (see for example [19, p.283]).

A local Artinian principal ideal ring is called a *special principal ideal ring*. It has only finitely many ideals, each of which is a power of the maximal ideal. For any ideal J of R , denote by $\mathbb{I}(J)$ the set $\{I : I \text{ an ideal of } R \text{ and } I \subseteq J\}$.

Lemma 3.1.2 [6, Lemma 2.3] *Let (R, \mathfrak{m}) be a local ring with $\mathfrak{m}^t = (0)$. If for a positive integer n , $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$ and \mathfrak{m}^n is a finitely generated R -module, then $\mathbb{I}(\mathfrak{m}^n) = \{\mathfrak{m}^i : n \leq i \leq t\}$. Moreover, if $n = 1$, then R is a special principal ideal ring.*

Lemma 3.1.3 *Let (R, \mathfrak{m}) be a local Artinian ring. If $I \in \mathbb{I}(\mathfrak{m}^{n-1}) \setminus \mathbb{I}(\mathfrak{m}^n)$ for some positive integer n , is a nonzero principal ideal, then $|\mathbb{I}(I)| = |\mathbb{I}(I \cap \mathfrak{m}^n)| + 1$.*

Proof. Since $I \in \mathbb{I}(\mathfrak{m}^{n-1}) \setminus \mathbb{I}(\mathfrak{m}^n)$ is a nonzero principal ideal, there exists $x \in \mathfrak{m}^{n-1} \setminus \mathfrak{m}^n$ such that $I = Rx$. Let $J \in \mathbb{I}(Rx)$ such that $J \neq I$ and $y \in J$. Thus $y = rx$ for some $r \in R$. If $r \notin \mathfrak{m}$, then r is an invertible element and so $Ry = Rx$, yielding a contradiction. Thus we have $r \in \mathfrak{m}$, so $y = rx \in \mathfrak{m}^n$. Therefore, $J \in \mathbb{I}(\mathfrak{m}^n)$. Hence $|\mathbb{I}(I)| = |\mathbb{I}(I \cap \mathfrak{m}^n)| + 1$. \square

Lemma 3.1.4 *Let (R, \mathfrak{m}) be a local Artinian ring. If I is a principal ideal such that $|\mathbb{I}(I)| = 3$, then $\mathfrak{m}^2 \subseteq \text{Ann}(I)$.*

Proof. Since I is a principal ideal, $I = Rx$ for some $x \in R$. Since $Rx \cong R/\text{Ann}(x)$ and Rx has only one nonzero proper R -submodule, $\mathfrak{m}/\text{Ann}(x)$ is the only nonzero proper ideal of $R/\text{Ann}(x)$. If $\mathfrak{m}^2 \not\subseteq \text{Ann}(x)$, then $\text{Ann}(x) + \mathfrak{m}^2 = \mathfrak{m}$, and by Nakayama's lemma, $\text{Ann}(x) = \mathfrak{m}$, yielding a contradiction. Thus $\mathfrak{m}^2 \subseteq \text{Ann}(x) = \text{Ann}(I)$. \square

3.2 Main Results

In [6], it was proved that if R is an Artinian Commutative ring and $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then either R has only finitely many ideals or R is a Gorenstein ring with $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$. We now improve the above result substantially by showing that the second situation is impossible (when $0 < \gamma(\mathbb{A}\mathbb{G}(R))$). Then

we characterize commutative Artinian rings whose annihilating-ideal graphs have finite genus (including genus zero).

Proposition 3.2.1 *Let R be a commutative Artinian ring with $0 < \gamma(\mathbb{AG}(R)) < \infty$. Then R has only finitely many ideals.*

Proof. Let R be a commutative Artinian ring and assume R has infinitely many ideals. Theorem 1.3.4 says that $R \cong R_1 \times \cdots \times R_n$ for some positive integer n , where each R_i ($1 \leq i \leq n$) is an Artinian local ring and the addition and the multiplication in the product are defined componentwise. Assume that $n \geq 2$. If $|\mathbb{I}(R)| = \infty$, then there exists R_i such that $|\mathbb{I}(R_i)| = \infty$. Without loss of generality, we can assume that R_1 has infinitely many ideals. Let $I_1 = (0 \times R_2 \times 0 \times \cdots \times 0)$, $I_2 = (\text{Ann}(\mathfrak{m}_1) \times 0 \times \cdots \times 0)$ and $I_3 = (\text{Ann}(\mathfrak{m}_1) \times R_2 \times 0 \times \cdots \times 0)$. Then for every proper ideal J of R_1 , we have $I_i(J \times 0 \times \cdots \times 0) = (0)$. Therefore, $K_{|\mathbb{I}(\mathfrak{m}_1)|-1,3}$ is a subgraph of $\mathbb{AG}(R)$. Since $|\mathbb{I}(\mathfrak{m}_1)| = \infty$, by the formula for the genus of complete bipartite graphs (Formula (1.2)), $\gamma(\mathbb{AG}(R)) = \infty$, yielding a contradiction. Thus, each R_i has only finitely many ideals, and therefore, R has only finitely many ideals. So we may assume that R is a local ring. If R is a field, then R has only finitely many ideals. Thus we can assume that R is not a field. If $|R/\mathfrak{m}| < \infty$, then one can easily see that R is a finite ring, so R has only finitely many ideals. Thus we can assume that $|R/\mathfrak{m}| = \infty$. If $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) \geq 2$, then $|\mathbb{I}(\text{Ann}(\mathfrak{m}))| = |\mathbb{I}(\mathfrak{m})| = \infty$ and since $\mathfrak{m}\text{Ann}(\mathfrak{m}) = (0)$, $K_{|\mathbb{I}(\mathfrak{m})|-4,3}$ is a subgraph of $\mathbb{AG}(R)$. Hence, by Formula (1.2), $\lceil (|\mathbb{I}(\mathfrak{m})| - 6)/4 \rceil \leq g$, and so

$|\mathbb{I}(\mathfrak{m})| \leq 4g + 6$, yielding a contradiction. Therefore, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) = 1$. Thus we can assume that $|R/\mathfrak{m}| = \infty$, $\gamma(\mathbb{A}\mathbb{G}(R)) = g$ for an integer $g > 0$, and $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) = 1$. Since R is an Artinian ring, there exists a positive integer t such that $\mathfrak{m}^{t+1} = (0)$ and $\mathfrak{m}^t \neq (0)$. Note that $\mathfrak{m}^t \subseteq \text{Ann}(\mathfrak{m})$ and $\dim(\text{Ann}(\mathfrak{m})) = 1$, thus $\text{Ann}(\mathfrak{m}) = \mathfrak{m}^t$. Let I be a minimal ideal of R . Then $I\mathfrak{m} = (0)$. Hence $I \subseteq \text{Ann}(\mathfrak{m})$ and so $I = \mathfrak{m}^t$. Therefore, \mathfrak{m}^t is the **unique minimal ideal** of R . We now proceed the proof using case by case analysis.

Case 1: $t = 1$, i.e., $\mathfrak{m}^2 = (0)$. Then since $\mathfrak{m} = \mathfrak{m}^t$ is also the unique minimal ideal, R has exactly two proper ideals as desired.

Case 2: $t = 2$, i.e., $\mathfrak{m}^2 \neq (0)$ and $\mathfrak{m}^3 = (0)$. We will prove that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 1$. Suppose on the contrary that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) > 1$. First, we assume that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) > 3$. Suppose that $\{x_1 + \mathfrak{m}^2, x_2 + \mathfrak{m}^2, x_3 + \mathfrak{m}^2, \dots, x_n + \mathfrak{m}^2\}$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$ over R/\mathfrak{m} . Since $Rx_1 \cong R/\text{Ann}(x_1)$ and by Lemma 3.1.3, Rx_1 has one nonzero proper R -submodule, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_1)/\mathfrak{m}^2) = n - 1$. Similarly, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_2)/\mathfrak{m}^2) = n - 1$. Therefore, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_2) \cap \text{Ann}(x_1))/\mathfrak{m}^2 = n - 2$ and since $n > 3$, $|\mathbb{I}(\text{Ann}(x_2) \cap \text{Ann}(x_1))| = \infty$. Since $(Rx_1)(\text{Ann}(x_2) \cap \text{Ann}(x_1)) = (0)$, $(Rx_2)(\text{Ann}(x_2) \cap \text{Ann}(x_1)) = (0)$, and $\mathfrak{m}^2(\text{Ann}(x_2) \cap \text{Ann}(x_1)) = (0)$, we conclude that $K_{|\mathbb{I}(\text{Ann}(x_2) \cap \text{Ann}(x_1))|, 3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Thus by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction.

Next assume that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 3$. Then there exist infinitely many subspaces with dimension one. Let Rx/\mathfrak{m}^2 and Ry/\mathfrak{m}^2 be two distinct subspaces of dimension one of $\mathfrak{m}/\mathfrak{m}^2$. Since $Rx \cong R/\text{Ann}(x)$ and Rx has only one nonzero proper R -submodule, $\mathfrak{m}/\text{Ann}(x)$ is the only nonzero proper ideal of $R/\text{Ann}(x)$. Therefore, $\text{v.dim}_{R/\mathfrak{m}}\text{Ann}(x)/\mathfrak{m}^2 = 2$. Similarly, $\text{v.dim}_{R/\mathfrak{m}}\text{Ann}(y)/\mathfrak{m}^2 = 2$. Therefore, $|\mathbb{I}(\text{Ann}(x))| = |\mathbb{I}(\text{Ann}(y))| = \infty$. If $\text{Ann}(x) = \text{Ann}(y)$, then since $(Rx)\text{Ann}(x) = (0)$, $(Ry)\text{Ann}(x) = (0)$, and $\mathfrak{m}^2\text{Ann}(x) = (0)$, we conclude that $K_{|\mathbb{I}(\text{Ann}(x))|,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Thus by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction. Thus we may assume that $\text{Ann}(x) \neq \text{Ann}(y)$. Since $\mathfrak{m}^2 \subseteq \text{Ann}(x) \cap \text{Ann}(y)$, then $(\text{Ann}(x) \cap \text{Ann}(y)) \neq (0)$ so $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x) \cap \text{Ann}(y))/\mathfrak{m}^2 = 1$. Suppose that $(Rx)(Ry) \neq (0)$. Since $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x) \cap \text{Ann}(y))/\mathfrak{m}^2 = 1$, there exists an ideal K such that $Rx - K - Ry$. Let I_1 be an ideal such that $I_1 \in \mathbb{I}(\text{Ann}(x)) \setminus \{K, Ry\}$ and $\text{v.dim}_{R/\mathfrak{m}}I_1/\mathfrak{m}^2 = 1$. Let J_1 be an ideal such that $J_1 \in \mathbb{I}(\text{Ann}(y)) \setminus \{Rx, Ry, K, I_1, \text{Ann}(x) \cap \text{Ann}(y)\}$ such that $\text{v.dim}_{R/\mathfrak{m}}J_1/\mathfrak{m}^2 = 1$. Let $K_1 = \text{Ann}(I_1) \cap \text{Ann}(J_1)$. Therefore, $Rx - I_1 - K_1 - J_1 - Ry$ is a path between Rx and Ry . Now, let $I_n \in \mathbb{I}(\text{Ann}(x)) \setminus \{I_i, J_i, K_i, Rx, Ry, \text{Ann}(x) \cap \text{Ann}(y), \text{Ann}(x) \cap \text{Ann}(K_i), i = 1, 2, \dots, n-1\}$ such that $\text{v.dim}_{R/\mathfrak{m}}I_n/\mathfrak{m}^2 = 1$ and $J_n \in \mathbb{I}(\text{Ann}(y)) \setminus \{I_i, J_{i-1}, K_{i-1}, I_n, Rx, Ry, \text{Ann}(x) \cap \text{Ann}(y), \text{Ann}(x) \cap \text{Ann}(K_{i-1}), i = 1, 2, \dots, n\}$ such that $\text{v.dim}_{R/\mathfrak{m}}J_n/\mathfrak{m}^2 = 1$. Suppose that $K_n = \text{Ann}(I_n) \cap \text{Ann}(J_n)$; then $Rx - I_n - K_n - J_n - Ry$ is a path between Rx and Ry . Therefore, there exist infinitely many paths between Rx and Ry . Thus either $(Rx)(Ry) = (0)$ or there exist infinitely many paths

between Rx and Ry . Since Rx/\mathfrak{m}^2 and Ry/\mathfrak{m}^2 are two arbitrary distinct one dimensional R/\mathfrak{m} -subspaces of $\text{Ann}(x)$ and there are infinitely many subspaces of dimension one of $\text{Ann}(x)$, one can easily see that $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction.

Next we assume that $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$. Let $I \neq \mathfrak{m}^2$ be an ideal. If $I \neq \mathfrak{m}$, then since $\mathfrak{m}^2 \subseteq I$ and $\text{v.dim}_{R/\mathfrak{m}} I/\mathfrak{m}^2 = 1$, we conclude that $I = Rx$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Note that $Rx \cong R/\text{Ann}(x)$. Since $|\mathbb{I}(Rx)^*| = 2$ (by Lemma 3.1.3), we conclude that there is exactly one ideal between $\text{Ann}(x)$ and R . Therefore, $\text{v.dim}_{R/\mathfrak{m}} \text{Ann}(x)/\mathfrak{m}^2 = 1$. So, by Lemma 3.1.3 $|\mathbb{I}(\text{Ann}(x))^*| = 2$. We conclude that every ideal except \mathfrak{m}^2 has degree at most 2, so $\mathbb{A}\mathbb{G}(R)$ is a subgraph of Figure 3.1, hence $\gamma(\mathbb{A}\mathbb{G}(R)) = 0$, yielding a contradiction. Therefore $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$. Thus by Lemma 3.1.2, R has only two nonzero proper ideals.

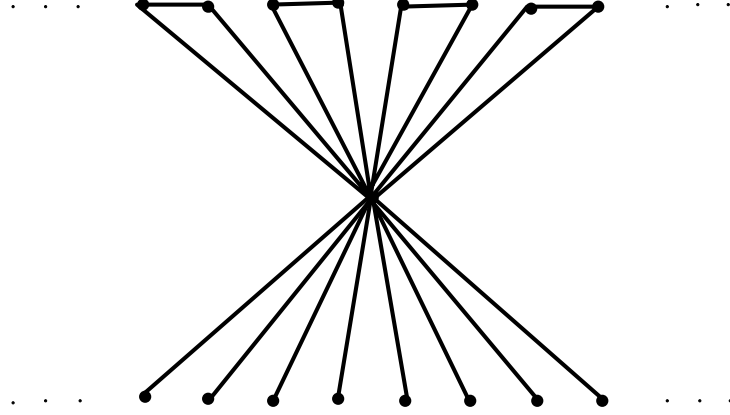


Figure 3.1: An infinite planar graph, which contains every infinite planar annihilating-ideal graph of a local ring with maximal ideal \mathfrak{m} such that $\mathfrak{m}^2 \neq (0)$ and $\mathfrak{m}^3 = (0)$.

Case 3: $t = 3$, i.e., $\mathfrak{m}^4 = (0)$. By [6, Theorem 2.5], $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \leq 2$. First assume that $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$. If $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 \geq 2$, then since $\mathfrak{m}^4 = (0)$, we conclude that $K_{|\mathbb{I}(\mathfrak{m}^2)|-1}$ is a subgraph of $\mathbb{AG}(R)$. Note that $|\mathbb{I}(\mathfrak{m}^2)| = \infty$. Thus by Formula (1.1), $\gamma(\mathbb{AG}(R)) = \infty$, yielding a contradiction. Therefore, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 = 1$, and so by Lemma 3.1.2, $\mathbb{I}(\mathfrak{m}^2)^* = \{\mathfrak{m}^2, \mathfrak{m}^3\}$.

We now claim that $|\mathbb{I}(\text{Ann}(\mathfrak{m}^2))| = \infty$. Let $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\mathfrak{m}^2 \not\subseteq Rx_1$. Then by Lemma 3.1.3, $|\mathbb{I}(Rx_1)| = |\mathbb{I}(\mathfrak{m}^2 \cap Rx_1)| + 1$. Thus $|\mathbb{I}(Rx_1)| = 3$, and by Lemma 3.1.4, $\mathfrak{m}^2 \subseteq \text{Ann}(x_1)$. Therefore, $\mathfrak{m}^2 \subseteq Rx_1$ or $\mathfrak{m}^2 \subseteq \text{Ann}(x_1)$. Let $x_2 \in \text{Ann}(x_1)$. Then $\mathfrak{m}^2 \subseteq \text{Ann}(x_1)$ or $\mathfrak{m}^2 \subseteq \text{Ann}(x_2)$. Let $x_{2i-1} \in \mathfrak{m} \setminus \mathfrak{m}^2$ ($i \geq 2$) such that $\{x_{2i-1} + \mathfrak{m}^2, x_2 + \mathfrak{m}^2\}$ and $\{x_{2i-1} + \mathfrak{m}^2, x_{2j-1} + \mathfrak{m}^2\}$

for $j = 1, 2, \dots, i-1$ be a basis for $\mathfrak{m}/\mathfrak{m}^2$. If $\mathfrak{m}^2 \not\subseteq Rx_{2i-1}$, then by Lemma 3.1.3, $|\mathbb{I}(Rx_{2i-1})| = 3$, so by Lemma 3.1.4, $\mathfrak{m}^2 \subseteq \text{Ann}(x_{2i-1})$. Since either $\mathfrak{m}^2 \subseteq \text{Ann}(x_1)$ or $\mathfrak{m}^2 \subseteq \text{Ann}(x_2)$, we conclude that either $(0) = \mathfrak{m}^2(Rx_1 + Rx_{2i-1}) = \mathfrak{m}^2\mathfrak{m} = \mathfrak{m}^3$ or $(0) = \mathfrak{m}^2(Rx_2 + Rx_{2i-1}) = \mathfrak{m}^2\mathfrak{m} = \mathfrak{m}^3$, yielding a contradiction. Thus $\mathfrak{m}^2 \subseteq Rx_{2i-1}$. Since $\mathfrak{m}x_{2i-1} \subseteq \mathfrak{m}^2$ and $|\mathbb{I}(\mathfrak{m}^2)| < \infty$, $\mathbb{I}(\mathfrak{m}x_{2i-1}) < \infty$. Note that $\mathfrak{m}x_{2i-1} \cong \mathfrak{m}/\text{Ann}(x_{2i-1})$ and $\mathbb{I}(\mathfrak{m}x_{2i-1}) < \infty$. Thus there are finitely many ideals between $\text{Ann}(x_{2i-1})$ and \mathfrak{m} . So $\dim(\text{Ann}(x_{2i-1}) + \mathfrak{m}^2)/\mathfrak{m}^2 = 1$. Hence $\text{Ann}(x_{2i-1}) \cap (\mathfrak{m} \setminus \mathfrak{m}^2) \neq (0)$. We can find $x_{2i} \in \text{Ann}(x_{2i-1}) \cap (\mathfrak{m} \setminus \mathfrak{m}^2)$. Since $\mathfrak{m}^2 \subseteq Rx_{2i-1}$, $\mathfrak{m}^2 \subseteq \text{Ann}(x_{2i})$. If $Rx_{2i} = Rx_{2j}$ for some $j = 1, 2, \dots, i-1$, then $Rx_{2i}(Rx_{2i-1} + Rx_{2j-1}) = 0$, and so $Rx_{2i}\mathfrak{m} = 0$, yielding a contradiction since $\text{Ann}(\mathfrak{m}) = \mathfrak{m}^t$. Since $Rx_{2i} \in \mathbb{I}(\text{Ann}(\mathfrak{m}^2))$ for each i , so $|\mathbb{I}(\text{Ann}(\mathfrak{m}^2))| = \infty$, as we claimed. Let $I \neq \mathfrak{m}$ be an ideal such that $I \not\subseteq \mathbb{I}(\mathfrak{m}^2)$. Next we divide our proof into following two subcases:

Subcase 3.1: $\mathfrak{m}^2 \not\subseteq I$. Let $y \in I \setminus \mathfrak{m}^2$. Let $x \in I \setminus Ry \cup \mathfrak{m}^2$. Since $\dim_{R/\mathfrak{m}}(I/\mathfrak{m}^2) = 1$, $Rx + \mathfrak{m}^2 = Ry + \mathfrak{m}^2$, so there exists $r \in R$ such that $x - ry \in \mathfrak{m}^2$. If $x - ry \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$, then since $|\mathbb{I}(\mathfrak{m}^2)| = 3$, we conclude that $R(x - ry) = \mathfrak{m}^2$. So $\mathfrak{m}^2 \subseteq I$, yielding a contradiction. Therefore, $x - ry \in \mathfrak{m}^3 \subseteq Ry$ and so $x \in Ry$, yielding a contradiction. Hence $I = Ry$. Note that $\mathfrak{m}^2 \not\subseteq Ry$. Thus by Lemma 3.1.3, $|\mathbb{I}(Ry)| = 3$. By Lemma 3.1.4, $\mathfrak{m}^2 \subseteq \text{Ann}(y)$. Since $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(y)/\mathfrak{m}^2) = 1$, there exists $z \in \text{Ann}(y) \setminus \mathfrak{m}^2$ such that $\text{Ann}(y) = Rz + \mathfrak{m}^2$. If $\mathfrak{m}^2 \subseteq Rz$, we conclude that $\text{Ann}(y) = Rz$. So by Lemma 3.1.3, $|\mathbb{I}(\text{Ann}(y))^*| = 3$. Therefore, $\mathfrak{m}^2 \subseteq \text{Ann}(I)$ and

$\deg(I) = 3$. If $\mathfrak{m}^2 \not\subseteq Rz$, then $|\mathbb{I}(Rz)| = 3$ and by Lemma 3.1.4, $\mathfrak{m}^2 Rz = (0)$. Since $\text{Ann}(y) = Rz + \mathfrak{m}^2$ and $\mathfrak{m}^2 Rz = (0)$, we conclude that $\text{Ann}(y) = \text{Ann}(\mathfrak{m}^2)$. Thus $Ry\text{Ann}(\mathfrak{m}^2) = (0)$, $\mathfrak{m}^2\text{Ann}(\mathfrak{m}^2) = (0)$, and $\mathfrak{m}^3\text{Ann}(\mathfrak{m}^2) = (0)$. Therefore, $K_{|\text{Ann}(\mathfrak{m}^2)|,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$, and so by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction.

Subcase 3.2: $\mathfrak{m}^2 \subseteq I$. Since $\text{v.dim}_{R/\mathfrak{m}}(I/\mathfrak{m}^2) = 1$, there exists $z \in I \setminus \mathfrak{m}^2$ such that $I = Rz + \mathfrak{m}^2$. If $\mathfrak{m}^2 \subseteq Rz$, we conclude that $I = Rz$. If $\mathfrak{m}^2 \subseteq \text{Ann}(z)$, then there exists exactly one ideal between $\text{Ann}(z)$ and R . Since $Rz \cong R/\text{Ann}(z)$ and by Lemma 3.1.3, $|\mathbb{I}(Rz)| = 4$, we conclude that there exist two ideals between $\text{Ann}(z)$ and R , yielding a contradiction. Thus $\mathfrak{m}^2 \not\subseteq \text{Ann}(z)$. As in the proof of Subcase 3.1, $\text{Ann}(z)$ is a principal ideal and $|\mathbb{I}(\text{Ann}(z))| = 3$, so $\deg(I) = 2$. If $\mathfrak{m}^2 \not\subseteq Rz$, then by Lemma 3.1.3 and Lemma 3.1.4, $\mathfrak{m}^2 Rz = (0)$. Since $I = Rz + \mathfrak{m}^2$ and $\mathfrak{m}^2 Rz = (0)$, we conclude that $I = \text{Ann}(\mathfrak{m}^2)$. If $\deg(I) \geq 3$, then $K_{|\mathbb{I}(\text{Ann}(\mathfrak{m}^2))|,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Since $|\mathbb{I}(\text{Ann}(\mathfrak{m}^2))| = \infty$, by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction. Therefore, $\deg(I) = 2$. Since every ideal except \mathfrak{m}^2 and \mathfrak{m}^3 has degree at most 3, and every ideal with degree 3 is adjacent to \mathfrak{m}^2 and \mathfrak{m}^3 , $\gamma(\mathbb{A}\mathbb{G}(R))$ is a subgraph of the graph in Figure 3.2. Thus $\gamma(\mathbb{A}\mathbb{G}(R)) = 0$, yielding a contradiction.

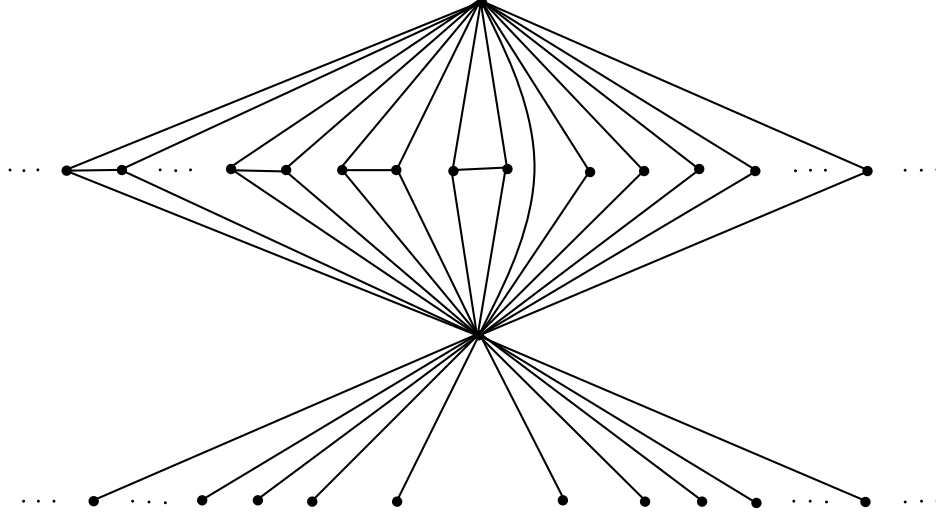


Figure 3.2: An infinite planar graph, which contains every infinite planar annihilating-ideal graph of a local ring with maximal ideal \mathfrak{m} such that $\mathfrak{m}^3 \neq (0)$ and $\mathfrak{m}^4 = (0)$.

Case 4: $t \geq 4$. Since $\mathfrak{m}^3\mathfrak{m}^t = \mathfrak{m}^3\mathfrak{m}^{t-1} = \mathfrak{m}^3\mathfrak{m}^{t-2} = (0)$, $K_{|\mathbb{I}(\mathfrak{m}^3)|-4,3}$ is a subgraph of $\mathbb{AG}(R)$. So by Formula (1.2), $\lceil (|\mathbb{I}(\mathfrak{m}^3)| - 6)/12 \rceil \leq g$. Hence, $|\mathbb{I}(\mathfrak{m}^3)| \leq 12g + 6$. If $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^{t-1}/\mathfrak{m}^t) \geq 2$, then Remark 3.1.1 implies that $|\mathbb{I}(\mathfrak{m}^{t-1})| = \infty$. Since $\mathfrak{m}^{t-1}\mathfrak{m}^{t-1} = (0)$ and $t \geq 3$, $K_{|\mathbb{I}(\mathfrak{m}^{t-1})|-1}$ is a subgraph of $\mathbb{AG}(R)$. Therefore, by Formula (1.1), $\gamma(\mathbb{AG}(R)) = \infty$, a contradiction. Thus $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^{t-1}/\mathfrak{m}^t) = 1$. Hence, by Lemma 3.1.2, there exists $x \in \mathfrak{m}^{t-1}$ such that $\mathfrak{m}^{t-1} = Rx$.

Now we prove that $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}/\mathfrak{m}^2 = 1$. Suppose on the contrary that $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}/\mathfrak{m}^2 = n \geq 2$. By Lemma 3.1.4, $\mathfrak{m}^2 \subseteq \text{Ann}(x)$, and since $\mathfrak{m}/\text{Ann}(x)$ is the only nonzero proper ideal of $R/\text{Ann}(x)$, $\text{v.dim}_{R/\mathfrak{m}}\text{Ann}(x)/\mathfrak{m}^2 = n - 1$.

Let $\{y_1 + \mathfrak{m}^2, y_2 + \mathfrak{m}^2, \dots, y_{n-1} + \mathfrak{m}^2, y + \mathfrak{m}^2\}$ be a basis for $\mathfrak{m}/\mathfrak{m}^2$ such that $\{y_1 + \mathfrak{m}^2, y_2 + \mathfrak{m}^2, \dots, y_{n-1} + \mathfrak{m}^2\}$ is a basis for $\text{Ann}(x)/\mathfrak{m}^2$. Since \mathfrak{m}^t is the only minimal ideal of R , $\mathfrak{m}^t \subseteq Ry$ and $|\mathbb{I}(Ry)| \geq 3$. If $|\mathbb{I}(Ry)| = 3$, then by Lemma 3.1.4, $\mathfrak{m}^2 \subseteq \text{Ann}(y)$. So $\mathfrak{m}^{t-1} \subseteq \text{Ann}(y)$. Hence, $\mathfrak{m}^{t-1}\mathfrak{m} = (0)$ ($\mathfrak{m}^{t-1}(Ry) = (0)$ and $\mathfrak{m}^{t-1}(Ry_1 + \dots + Ry_{n-1}) = (0)$), a contradiction. Therefore, $|\mathbb{I}(Ry)| \geq 4$. We now divide the proof into two subcases according to whether or not $|\mathbb{I}(Ry)| = \infty$, and show that both cases are impossible.

Subcase 4.1: $4 \leq |\mathbb{I}(Ry)| < \infty$. Since $Ry \cong R/\text{Ann}(y)$, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(y) + \mathfrak{m}^2)/\mathfrak{m}^2 = n - 1$. If $n \geq 3$, then $|\mathbb{I}(\text{Ann}(y))| = \infty$ and $K_{|\mathbb{I}(\text{Ann}(y))|, |\mathbb{I}(Ry)|}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. So by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction. Thus we have $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}/\mathfrak{m}^2 = n = 2$. We now claim that $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}^2/\mathfrak{m}^3 = 1$. Suppose that $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}^2/\mathfrak{m}^3 = l \geq 3$. Since $\mathfrak{m}^2y \cong \mathfrak{m}^2/\text{Ann}(y) \cap \mathfrak{m}^2$ and $|\mathbb{I}(\mathfrak{m}^2y)| \leq |\mathbb{I}(\mathfrak{m}^3)| < \infty$, we conclude that $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(y) \cap \mathfrak{m}^2 + \mathfrak{m}^3/\mathfrak{m}^3) = l - 1$, and so $|\mathbb{I}(\text{Ann}(y))| = \infty$. Since $K_{|\mathbb{I}(\text{Ann}(y))|, |\mathbb{I}(Ry)|}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$, by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction. So we have $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}^2/\mathfrak{m}^3 \leq 2$.

Assume that $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}^2/\mathfrak{m}^3 = 2$. Let $x_1 \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$. Assume that $\mathfrak{m}^{t-2} \not\subseteq Rx_1$. By Lemma 3.1.3, $|\mathbb{I}(Rx_1)| = |\mathbb{I}(\mathfrak{m}^3 \cap Rx_1)| + 1$. Since $\mathfrak{m}^{t-2} \not\subseteq Rx_1$, either $\mathfrak{m}^3 \cap Rx_1 = \mathfrak{m}^t$ or $\mathfrak{m}^3 \cap Rx_1 = \mathfrak{m}^{t-1}$. We conclude that $|\mathbb{I}(Rx_1)|$ is either 3 or 4. If $|\mathbb{I}(Rx_1)| = 3$, then by Lemma 3.1.4, $\mathfrak{m}^2 \subseteq \text{Ann}(x_1)$, and so $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$. Assume that $|\mathbb{I}(Rx_1)| = 4$. Since $Rx_1 \cong R/\text{Ann}(x_1)$ and Rx_1 has only two nonzero proper ideals, \mathfrak{m} and $\text{Ann}(x_1) + \mathfrak{m}^2$ are the only nonzero proper ideals of $R/\text{Ann}(x_1)$. Therefore, $\text{Ann}(x_1) + \mathfrak{m}^2 = \text{Ann}(x_1) + \mathfrak{m}^3$, and so

$(\text{Ann}(x_1) + \mathfrak{m}^2)\mathfrak{m}^{t-4} = (\text{Ann}(x_1) + \mathfrak{m}^3)\mathfrak{m}^{t-4}$. Thus $\text{Ann}(x_1)\mathfrak{m}^{t-4} + \mathfrak{m}^{t-2} = \text{Ann}(x_1)\mathfrak{m}^{t-4} + \mathfrak{m}^{t-1} \subseteq \text{Ann}(x_1)$. Therefore, $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$. So, in both cases, $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$. Thus if $\mathfrak{m}^{t-2} \not\subseteq Rx_1$, then $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$. Assume that $\mathfrak{m}^{t-2} \subseteq Rx_1$. Note that $\mathfrak{m}^2x_1 \cong \mathfrak{m}^2/(\text{Ann}(x_1) \cap \mathfrak{m}^2)$. Since $\mathfrak{m}^2x_1 \subseteq \mathfrak{m}^3$ and $|\mathbb{I}(\mathfrak{m}^3)| < \infty$, $\text{v.dim}_{R/\mathfrak{m}}((\text{Ann}(x_1) \cap \mathfrak{m}^2) + \mathfrak{m}^3)/\mathfrak{m}^3 = 1$, so $\text{Ann}(x_1) \cap (\mathfrak{m}^2 \setminus \mathfrak{m}^3) \neq (0)$. Let $x_2 \in \text{Ann}(x_1) \cap (\mathfrak{m}^2 \setminus \mathfrak{m}^3)$. Then $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_2)$ since $\mathfrak{m}^{t-2} \subseteq Rx_1$. Therefore, $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$ or $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_2)$.

Let $x_{2i-1} \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$ such that $\{x_{2i-1} + \mathfrak{m}^3, x_2 + \mathfrak{m}^3\}$ and $\{x_{2i-1} + \mathfrak{m}^3, x_{2j-1} + \mathfrak{m}^3\}$ for $j = 1, 2, \dots, i-1$, be basis for $\mathfrak{m}^2/\mathfrak{m}^3$. Suppose that $\mathfrak{m}^{t-2} \not\subseteq Rx_{2i-1}$. Therefore, by Lemma 3.1.3, $|\mathbb{I}(Rx_{2i-1})| = |\mathbb{I}(\mathfrak{m}^3 \cap Rx_{2i-1})| + 1$. Since $\mathfrak{m}^{t-2} \not\subseteq Rx_{2i-1}$, $\mathfrak{m}^3 \cap Rx_{2i-1} = \mathfrak{m}^t$ or $\mathfrak{m}^3 \cap Rx_{2i-1} = \mathfrak{m}^{t-1}$. We conclude that $|\mathbb{I}(Rx_{2i-1})|$ is either 3 or 4. If $|\mathbb{I}(Rx_{2i-1})| = 3$, then by Lemma 3.1.4, $\mathfrak{m}^2 \subseteq \text{Ann}(x_{2i-1})$, and so $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_{2i-1})$. Assume that $|\mathbb{I}(Rx_{2i-1})| = 4$. Since $Rx_{2i-1} \cong R/\text{Ann}(x_{2i-1})$ and Rx_{2i-1} has only two nonzero proper ideals, \mathfrak{m} and $\text{Ann}(x_{2i-1}) + \mathfrak{m}^2$ are the only nonzero proper ideals of $R/\text{Ann}(x_{2i-1})$. Therefore, $\text{Ann}(x_{2i-1}) + \mathfrak{m}^2 = \text{Ann}(x_{2i-1}) + \mathfrak{m}^3$, and so $(\text{Ann}(x_{2i-1}) + \mathfrak{m}^2)\mathfrak{m}^{t-4} = (\text{Ann}(x_{2i-1}) + \mathfrak{m}^3)\mathfrak{m}^{t-4}$. Thus $\text{Ann}(x_{2i-1})\mathfrak{m}^{t-4} + \mathfrak{m}^{t-2} = \text{Ann}(x_{2i-1})\mathfrak{m}^{t-4} + \mathfrak{m}^{t-1} \subseteq \text{Ann}(x_{2i-1})$. Therefore, $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_{2i-1})$. So either $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$ or $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_2)$, we conclude that either $(0) = \mathfrak{m}^{t-2}(Rx_1 + Rx_{2i-1}) = \mathfrak{m}^{t-2}\mathfrak{m}^2 = \mathfrak{m}^t = 0$ or $(0) = \mathfrak{m}^{t-2}(Rx_2 + Rx_{2i-1}) = \mathfrak{m}^{t-2}\mathfrak{m}^2 = \mathfrak{m}^t = 0$, yielding a contradiction. Thus $\mathfrak{m}^{t-2} \subseteq Rx_{2i-1}$. Since $\mathfrak{m}^2x_{2i-1} \subseteq \mathfrak{m}^3$ and $|\mathbb{I}(\mathfrak{m}^3)| < \infty$, $|\mathbb{I}(\mathfrak{m}^2x_{2i-1})| < \infty$. Note that $\mathfrak{m}^2x_{2i-1} \cong \mathfrak{m}^2/(\text{Ann}(x_{2i-1}) \cap \mathfrak{m}^2)$. Thus there are finitely many ideals

between $\text{Ann}(x_{2i-1}) \cap \mathfrak{m}^2$ and \mathfrak{m}^2 . So $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_{2i-1}) \cap \mathfrak{m}^2 + \mathfrak{m}^3)/\mathfrak{m}^3 = 1$, hence $\text{Ann}(x_{2i-1}) \cap (\mathfrak{m}^2 \setminus \mathfrak{m}^3) \neq (0)$. Therefore, we can find $x_{2i} \in \text{Ann}(x_{2i-1}) \cap (\mathfrak{m}^2 \setminus \mathfrak{m}^3)$ and so $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_{2i})$. If $Rx_{2i} = Rx_{2j}$ for some $j = 1, 2, \dots, i-1$, then $Rx_{2i}(Rx_{2i-1} + Rx_{2j-1}) = 0$, and so $Rx_{2i}\mathfrak{m}^2 = 0$. Since $\mathfrak{m}^t\mathfrak{m}^2 = \mathfrak{m}^{t-1}\mathfrak{m}^2 = (Rx_{2i-1})\mathfrak{m}^2 = (0)$, $K_{|\mathbb{I}(\mathfrak{m}^2)|-4,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$ and by Formula (1.2) $\gamma(\Gamma(R)) = \infty$, yielding a contradiction. Therefore, $Rx_{2i} \in \mathbb{I}(\text{Ann}(\mathfrak{m}^{t-2}))$ for each i . Thus $|\mathbb{I}(\text{Ann}(\mathfrak{m}^{t-2}))| = \infty$. Since $\mathfrak{m}^t\text{Ann}(\mathfrak{m}^{t-2}) = \mathfrak{m}^{t-1}\text{Ann}(\mathfrak{m}^{t-2}) = \mathfrak{m}^{t-2}\text{Ann}(\mathfrak{m}^{t-2}) = (0)$, $K_{|\mathbb{I}(\text{Ann}(\mathfrak{m}^{t-2}))|,3}$ is a subgraph in $\mathbb{A}\mathbb{G}(R)$, and so $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction. Therefore, $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}^2/\mathfrak{m}^3 = 1$, as claimed.

Recall that $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}/\mathfrak{m}^2 = 2$. Let $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$. Assume that $\mathfrak{m}^{t-2} \not\subseteq Rx_1$. By Lemma 3.1.3, $|\mathbb{I}(Rx_1)| = |\mathbb{I}(\mathfrak{m}^2 \cap Rx_1)| + 1$. Since $\mathfrak{m}^{t-2} \not\subseteq Rx_1$, either $\mathfrak{m}^2 \cap Rx_1 = \mathfrak{m}^t$ or $\mathfrak{m}^2 \cap Rx_1 = \mathfrak{m}^{t-1}$. We conclude that either $|\mathbb{I}(Rx_1)| = 3$ or $|\mathbb{I}(Rx_1)| = 4$. If $|\mathbb{I}(Rx_1)| = 3$, then by Lemma 3.1.4, $\mathfrak{m}^2 \subseteq \text{Ann}(x_1)$, and so $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$. Assume that $|\mathbb{I}(Rx_1)| = 4$. Since $Rx_1 \cong R/\text{Ann}(x_1)$ and Rx_1 has only two nonzero proper ideals, \mathfrak{m} and $\text{Ann}(x_1) + \mathfrak{m}^2$ are the only nonzero proper ideals of $R/\text{Ann}(x_1)$. Therefore, $\text{Ann}(x_1) + \mathfrak{m}^2 = \text{Ann}(x_1) + \mathfrak{m}^3$, and so $(\text{Ann}(x_1) + \mathfrak{m}^2)\mathfrak{m}^{t-3} = (\text{Ann}(x_1) + \mathfrak{m}^3)\mathfrak{m}^{t-3}$. Hence $\text{Ann}(x_1)\mathfrak{m}^{t-3} + \mathfrak{m}^{t-1} = \text{Ann}(x_1)\mathfrak{m}^{t-3} + \mathfrak{m}^t \subseteq \text{Ann}(x_1)$. Thus $\mathfrak{m}^{t-1} \subseteq \text{Ann}(x_1)$. Also, $(\text{Ann}(x_1) + \mathfrak{m}^2)\mathfrak{m}^{t-4} = (\text{Ann}(x_1) + \mathfrak{m}^3)\mathfrak{m}^{t-4}$. Thus $\text{Ann}(x_1)\mathfrak{m}^{t-4} + \mathfrak{m}^{t-2} = \text{Ann}(x_1)\mathfrak{m}^{t-4} + \mathfrak{m}^{t-1} \subseteq \text{Ann}(x_1)$. Therefore, $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$. Thus, if $\mathfrak{m}^{t-2} \not\subseteq Rx_1$, then $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$. Assume that $\mathfrak{m}^{t-2} \subseteq Rx_1$. Note that $\mathfrak{m}x_1 \cong \mathfrak{m}/\text{Ann}(x_1)$. Since $\mathfrak{m}x_1 \subseteq \mathfrak{m}^2$ and

$|\mathbb{I}(\mathfrak{m}^2)| < \infty$, there exist finitely many ideals between \mathfrak{m} and $\text{Ann}(x_1)$. Therefore, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_1) + \mathfrak{m}^2)/\mathfrak{m}^2 = 1$, so $\text{Ann}(x_1) \cap (\mathfrak{m} \setminus \mathfrak{m}^2) \neq (0)$. Let $x_2 \in \text{Ann}(x_1) \cap (\mathfrak{m} \setminus \mathfrak{m}^2)$. Then $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_2)$. Therefore, either $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$ or $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_2)$.

Let $x_{2i-1} \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\{x_{2i-1} + \mathfrak{m}^2, x_2 + \mathfrak{m}^2\}$ and $\{x_{2i-1} + \mathfrak{m}^2, x_{2j-1} + \mathfrak{m}^2\}$ for $j = 1, 2, \dots, i-1$ be a basis for $\mathfrak{m}/\mathfrak{m}^2$. As in the above, if $\mathfrak{m}^{t-2} \not\subseteq Rx_{2i-1}$, then $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_{2i-1})$. Since either $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_1)$ or $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_2)$, we conclude that either $(0) = \mathfrak{m}^{t-1}(Rx_1 + Rx_{2i-1}) = \mathfrak{m}^{t-1}\mathfrak{m} = \mathfrak{m}^t$ or $(0) = \mathfrak{m}^{t-1}(Rx_2 + Rx_{2i-1}) = \mathfrak{m}^{t-1}\mathfrak{m} = \mathfrak{m}^t$, yielding a contradiction. Thus $\mathfrak{m}^{t-2} \subseteq Rx_{2i-1}$. Since $\mathfrak{m}x_{2i-1} \subseteq \mathfrak{m}^2$ and $|\mathbb{I}(\mathfrak{m}^2)| < \infty$, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_{2i-1}) + \mathfrak{m}^2)/\mathfrak{m}^2 = 1$, so $\text{Ann}(x_{2i-1}) \cap (\mathfrak{m} \setminus \mathfrak{m}^2) \neq (0)$. Therefore, we can find $x_{2i} \in \text{Ann}(x_{2i-1}) \cap (\mathfrak{m} \setminus \mathfrak{m}^2)$ and so $\mathfrak{m}^{t-2} \subseteq \text{Ann}(x_{2i})$. If $Rx_{2i} = Rx_{2j}$ for some $j = 1, 2, \dots, i-1$, then $Rx_{2i}(Rx_{2i-1} + Rx_{2j-1}) = 0$, and so $Rx_{2i}\mathfrak{m} = 0$, yielding a contradiction. Since for every i , $Rx_{2i} \in \mathbb{I}(\text{Ann}(\mathfrak{m}^{t-2}))$, $|\mathbb{I}(\text{Ann}(\mathfrak{m}^{t-2}))| = \infty$. Note that $\mathfrak{m}^t(\text{Ann}(\mathfrak{m}^{t-2})) = \mathfrak{m}^{t-1}(\text{Ann}(\mathfrak{m}^{t-2})) = \mathfrak{m}^{t-2}(\text{Ann}(\mathfrak{m}^{t-2})) = (0)$. Thus $K_{|\text{Ann}(\mathfrak{m}^{t-2})|,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. So by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, yielding a contradiction.

Subcase 4.2: $|\mathbb{I}(Ry)| = \infty$. Suppose that $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}^2/\mathfrak{m}^3 \geq 2$. If $|\mathbb{I}(\text{Ann}(y))| \geq 4$, then $K_{|\mathbb{I}(Ry)|,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$ and so by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. We may assume that $|\mathbb{I}(\text{Ann}(y))| = 3$. Since $\text{Ann}(y) \not\subseteq \mathfrak{m}^3$ (since $\mathfrak{m}^2y \cong \mathfrak{m}^2/(\text{Ann}(y) \cap \mathfrak{m}^2)$), there exist only finitely many ideals between $\text{Ann}(y)$ and \mathfrak{m}^2 . Therefore, there exists $z \in \text{Ann}(y) \setminus \mathfrak{m}^3$. Since $|\mathbb{I}(\text{Ann}(y))| = 3$, $Rz = \text{Ann}(y)$. Thus by Lemma 3.1.4

$\mathfrak{m}^2 \subseteq \text{Ann}(z) = \text{Ann}(\text{Ann}(y))$. Since $\mathfrak{m}^2 \text{Ann}(y) = (0)$, $\mathfrak{m}^2 \mathfrak{m}^{t-1} = (0)$, and $\mathfrak{m}^t \mathfrak{m}^2 = (0)$, $K_{|\mathbb{I}(\mathfrak{m}^2)|,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Since $|\mathbb{I}(\mathfrak{m}^2)| = \infty$, by Formula (1.2), $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. Therefore, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 = 1$ and by Lemma 3.1.2, $|\mathbb{I}(\mathfrak{m}^2)| < \infty$. Also, by Lemma 3.1.3, $|\mathbb{I}(Ry)| < \infty$, yielding a contradiction.

Therefore, we always have $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$, and by Lemma 3.1.2, R has only finitely many ideals. The proof is complete. \square

As a consequence of Proposition 3.2.1, we obtain the following.

Theorem 3.2.2 *Let R be an Artinian ring with $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$. Then the following results hold.*

- (1) *If R is a non-local ring, then R has only finitely many ideals.*
- (2) *If R is a local ring with maximal ideal \mathfrak{m} such that $\mathfrak{m}^t \neq (0)$ and $\mathfrak{m}^{t+1} = (0)$, then we have the following:*
 - (a) *If $t = 1$, then R is either finite or a special principal ideal ring.*
 - (b) *If $t = 2$, then one of the following holds:*
 - (b.1) *R is finite;*
 - (b.2) *R is a special principal ideal ring;*
 - (b.3) *$\gamma(\mathbb{A}\mathbb{G}(R)) = 0$, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 = 1$, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$, R has infinitely many ideals, and $\mathbb{A}\mathbb{G}(R)$ is a subgraph of Figure 1.*
 - (c) *If $t = 3$, then one of the following holds:*
 - (c.1) *R is finite;*
 - (c.2) *R is a special principal ideal ring;*

(c.3) $\gamma(\mathbb{A}\mathbb{G}(R)) = 0$, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 = \text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^3/\mathfrak{m}^4 = 1$, R has infinitely many ideals, and $\mathbb{A}\mathbb{G}(R)$ is a subgraph of Figure 2.

(d) If $t \geq 4$, either R is finite or a special principal ideal ring.

Proof. Let R be an Artinian ring with $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$. If R is a non-local ring, then as in the proof of Proposition 3.2.1, R has only finitely many ideals, giving (1). If R is a local ring with maximal ideal \mathfrak{m} , then there exists positive integer t such that $\mathfrak{m}^t \neq (0)$ and $\mathfrak{m}^{t+1} = (0)$. If $|R/\mathfrak{m}| < \infty$, then one can easily check that R is finite. Now, we may assume that $|R/\mathfrak{m}| = \infty$. We have the following cases according to the value of t :

Case 1: $t = 1$, i.e., $\mathfrak{m}^2 = (0)$. Then by Case 1 in Proposition 3.2.1, R is a special principal ideal ring.

Case 2: $t = 2$, i.e., $\mathfrak{m}^3 = (0)$. By Case 2 in Proposition 3.2.1, either R is a special principal ideal ring or $\gamma(\mathbb{A}\mathbb{G}(R)) = 0$, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 = 1$, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$, R has infinitely many ideals, and $\mathbb{A}\mathbb{G}(R)$ is a subgraph of Figure 1.

Case 3: $t = 3$, i.e., $\mathfrak{m}^4 = (0)$. By Case 3 in Proposition 3.2.1, either R is a special principal ideal ring or $\gamma(\mathbb{A}\mathbb{G}(R)) = 0$, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 = \text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^3/\mathfrak{m}^4 = 1$, R has infinitely many ideals, and $\mathbb{A}\mathbb{G}(R)$ is a subgraph of Figure 2.

Case 4: $t \geq 4$. By Case 4 in Proposition 3.2.1, R is a special principal ideal ring.

Chapter 4

The Annihilating-Product-One Side-Ideal Graph

In this section we extend the definition of the annihilating-ideal graph to non-commutative rings. We introduce various ways to define the annihilating-ideal graph of a non-commutative ring. The first definition gives a directed graph denoted by $(APO)G(R)$. The other definition yields an undirected graph denoted by $\overline{(APO)}G(R)$. It is shown that $(APO)G(R)$ is not necessarily connected, but $\overline{(APO)}G(R)$ is connected and the diameter of $\overline{(APO)}G(R)$ is less than or equal to 3. Also, we show that if $(APO)G(R)$ has DCC (resp., ACC) on its vertices, then R is an Artinian (resp., Noetherian) ring. It is shown that $\overline{(APO)}G(R)$ has some features similar to that of the annihilating-ideal graph of a commutative ring. Finally, we investigate the diameter and the girth of square matrices over commutative rings

$M_{n \times n}(R)$, where $n \geq 2$. It is shown that $\text{diam}(\overline{(APO)}G(M_{n \times n}(R))) \geq 2$ and $\text{gr}(\overline{(APO)}G(M_{n \times n}(R))) = 3$, where $n \geq 2$.

We adopt the following notations in this chapter for a ring R ;

$$IPO(R) = \{A \subseteq R : A = IJ \text{ such that } I \text{ and } J \text{ are left or right ideals}\},$$

$$APO(R) = \{A \in IPO(R) \text{ and there exists } B \in (IPO(R))^* \text{ such that } AB = \{0\} \text{ or } BA = \{0\}\},$$

$$A_l(R) = \{\text{All left ideals in } APO(R)\},$$

$$A_r(R) = \{\text{All right ideals in } APO(R)\},$$

$$A^l(R) = \{A \in APO(R), \text{ where there exists } B \in APO(R)^* \text{ such that } BA = \{0\}\},$$

$$A^r(R) = \{A \in APO(R), \text{ where there exists } B \in APO(R)^* \text{ such that } AB = \{0\}\},$$

$$A^t(R) = \{A \in APO(R), \text{ where there exists left or right ideal } I \in (APO(R))^* \text{ such that } IA = \{0\} \text{ and } AII = \{0\}\},$$

$$I(R) = \{I \subseteq R : I \text{ is a left or right ideal}\}.$$

4.1 Directed Annihilating-Product-One Side-Ideal Graph

In this section we define a directed annihilating-product-one side-ideal graph of a ring denoted by $(APO)G(R)$. It is shown that $(APO)G(R)$ is not necessarily connected. We find necessary and sufficient conditions such that $(APO)G(R)$ is connected. Also, we show that if $(APO)G(R)$ has DCC

(resp., ACC) on its vertices, then R is an Artinian (resp., Noetherian) ring.

Let R be a ring with identity, I be a left or right ideal and J be a left or right ideal of R . We define $IJ = \sum_{i=1}^n a_i b_i$ such that $a_i \in I$ and $b_i \in J$ for $i = 1, 2, \dots, n$. It is easy to see this multiplication is associative. The following lemma shows that $IPO(R)$ with the above multiplication is a semigroup.

Lemma 4.1.1 *$IPO(R)$ is a semigroup.*

Proof. Let $A \in IPO(R)$ and $B \in IPO(R)$. Then there exist $I_1, J_1, I_2, J_2 \in I(R)$ such that $A = I_1 J_1$ and $B = I_2 J_2$. We show that $AB = (I_1 J_1)(I_2 J_2) \in IPO(R)$.

Case 1: J_1 is a left ideal. Then $AB = I_1(J_1 I_2 J_2) \in IPO(R)$.

Case 2: J_1 is a right ideal and either I_2 is a left ideal or J_2 is a right ideal. Then $AB = (I_1 J_1)(I_2 J_2) \in IPO(R)$.

Case 3: J_1 is a right ideal and, I_2 is a right ideal and J_2 is a left ideal. Then $AB = (I_1 J_1 I_2) J_2 \in IPO(R)$.

Therefore $AB \in IPO(R)$. Since the multiplication is associative, $IPO(R)$ is a semigroup. \square

We define a directed graph $(APO)G(R)$ with vertices $APO(R)^* = APO(R) \setminus \{0\}$, and $A \rightarrow B$ is an edge between distinct vertices A and B if $AB = \{0\}$. $(APO)G(R)$ is a directed graph on $APO(R)^*$. If R is a commutative ring, $A \rightarrow B$ is an edge whenever $B \rightarrow A$ is an edge. Therefore, for a commutative ring R if we view $(APO)G(R)$ as an undirected graph, this definition agrees with the usual definition of the annihilating-ideal graph of a commutative

ring.

We say that a directed graph Γ is connected if there is a path following the directed edges of Γ from any vertex of Γ to any other vertex of Γ . The next example show that, unlike the case for a commutative ring [15], $(APO)G(R)$ need not be connected if R is non-commutative.

Example 4.1.2 *Let K be a field and $V = \bigoplus_{i=1}^{\infty} K$. Then $R = HOM_K(V, V)$, under point-wise addition and multiplication taken to be composition functions, is an infinite non-commutative ring with identity. Let $\pi_1 : V \rightarrow V$ be defined by $(a_1, a_2, \dots) \mapsto (a_1, 0, \dots)$ and $f : V \rightarrow V$ be defined by $(a_1, a_2, \dots) \mapsto (0, a_1, a_2, \dots)$. Then $\pi_1, f \in R$. Note that $(R\pi_1)(fR) = \{0\}$, so $(APO)G(R) \neq \emptyset$. However $(APO)G(R)$ is not connected as there is no path leading from the vertex (fR) to any other vertex of $(APO)G(R)$ since there exists $g : V \rightarrow V$ be defined by $(a_1, a_2, \dots) \mapsto (a_2, a_3, \dots)$ and $g \in R$ such that $gf = 1_R$.*

Theorem 4.1.3 *Let R be a ring. Then $(APO)G(R)$ is connected if and only if $A^l(R) = A^r(R)$. Moreover, if $(APO)G(R)$ is connected, then $\text{diam}((APO)G(R)) \leq 3$.*

Proof. Suppose that $A^l(R) = A^r(R)$.

Let A and B be distinct vertices of $(APO)G(R)$. Then $A \neq \{0\}$ and $B \neq \{0\}$.

Case 1: $AB = \{0\}$. Then $A \rightarrow B$ is a path.

Case 2: $AB \neq \{0\}$, $A^2 = \{0\}$ and $B^2 = \{0\}$. Then $A \rightarrow AB \rightarrow B$ is a path.

Case 3: $AB \neq \{0\}$, $B^2 \neq \{0\}$ and $A^2 = \{0\}$. Then there exists $C \in APO(R) \setminus \{A, B, 0\}$ such that $CB = \{0\}$. If $AC = \{0\}$, then $A \rightarrow C \rightarrow B$ is a path. If $AC \neq \{0\}$, then $A \rightarrow AC \rightarrow B$ is a path.

Case 4: $AB \neq \{0\}$, $A^2 \neq \{0\}$ and $B^2 = \{0\}$. Then there exists $D \in APO(R) \setminus \{A, B, 0\}$ such that $AD = \{0\}$. If $DB = \{0\}$, then $A \rightarrow D \rightarrow B$ is a path. If $DB \neq \{0\}$, then $A \rightarrow DB \rightarrow B$ is a path.

Case 5: $AB \neq \{0\}$, $A^2 \neq \{0\}$ and $B^2 \neq \{0\}$. Then there exists $C \in APO(R) \setminus \{A, B, 0\}$ such that $AC = \{0\}$ and $D \in APO(R) \setminus \{A, B, 0\}$ such that $DB = \{0\}$ since $A^l(R) = A^r(R)$.

Subcase 5.1: $C = D$. Then $A \rightarrow C \rightarrow B$ is a path.

Subcase 5.2: $C \neq D$. If $CD = \{0\}$, then $A \rightarrow C \rightarrow D \rightarrow B$ is a path. If $CD \neq \{0\}$, then $A \rightarrow CD \rightarrow B$ is a path.

Thus $(APO)G(R)$ is connected and $\text{diam}((APO)G(R)) \leq 3$.

The converse follows by definitions. □

A Duo ring is a ring in which every one sided ideal is two sided.

Proposition 4.1.4 *Let R be an Artinian Duo ring. Then $A^l(R) = A^r(R) = IPO(R) \setminus \{R\}$.*

Proof. Let R be an Artinian Duo ring. Then by [23, Lemma 4.2] $R = (R_1, m_1) \times (R_2, m_2) \times \dots (R_n, m_n)$, where every R_i is a local Duo ring with unique maximal ideal m_i . Let $A \in IPO(R)$. Then $A = (I_1 \times I_2 \times \dots \times I_n)$

$(J_1 \times J_2 \times \dots \times J_n)$, where every $I_i, 1 \leq i \leq n$ and $J_j, 1 \leq j \leq n$, is an ideal. Since $A \neq R$, there exists I_i or J_j such that $I_i \neq R$ or $J_j \neq R$. Without loss of generality assume that $I_i \neq R$. Then $A = (I_1 \times I_2 \times \dots \times I_n)$
 $(J_1 \times J_2 \times \dots \times J_n) \subseteq (R_1 \times \dots \times I_i \times \dots \times R_n) (R_1 \times \dots \times R_i \times \dots \times R_n)$.
 Suppose k is the smallest positive integer such that $I_i^k = \{0\}$. Thus $(0 \times \dots \times I_i^{k-1} \times \dots \times 0)((R_1 \times \dots \times I_i \times \dots \times R_n)(R_1 \times \dots \times R_i \times \dots \times R_n)) = \{0\}$ and
 $((R_1 \times \dots \times I_i \times \dots \times R_n)(R_1 \times \dots \times R_i \times \dots \times R_n))(0 \times \dots \times I_i^{k-1} \times \dots \times 0) = \{0\}$.
 Therefore $A \in A^l(R)$ and $A \in A^r(R)$. Hence $IPO(R) \subseteq A^r$ and $IPO(R) \subseteq A^l(R)$. We conclude that $A^r(R) = IPO(R) = A^l(R)$. \square

The following corollary shows that for an Artinian Duo ring $(APO)G(R)$ is connected with $\text{diam}((APO)G(R)) \leq 3$.

Corollary 4.1.5 *Let R be an Artinian Duo ring. Then $(APO)G(R)$ is connected with $\text{diam}((APO)G(R)) \leq 3$.*

Proof. The result follows from Theorem 4.1.3 and Proposition 4.1.4. \square

It is well known that if $|Z(R)| \geq 2$ is finite then $|R|$ is finite. For any left or right ideal I , let $\text{ad}(I) = \{A \in APO(R)^* \text{ such that } I = A \text{ or } I \rightarrow A \text{ or there exists } B \in APO(R)^* \text{ such that } I \rightarrow B \rightarrow A\}$. We know that $|\text{ad}(I)| \subseteq Z(R)$. The following proposition shows that if I is a left or right principal ideal of R and all left and right ideals of $\text{ad}(I)$ are finite, then R is finite.

Proposition 4.1.6 *Let R be a ring and I be a left or right principal ideal of R . If all of left and right ideals of $\text{ad}(I)$ are finite, then R is finite.*

Proof. Without loss of generality assume that I is a left principal ideal.

Therefore $I = Rx$ for some $x \in R$.

Case 1: $I = Ann_r(x)$ and $Ann_r(x)Ann_l(x) = \{0\}$. Then

$$I \rightarrow Ann_l(x)$$

and $Ann_l(x) \in ad(I)$. Therefore $Ann_l(x)$ is finite. Since $I \cong R/Ann_l(x)$, $|R| = |I||Ann_l(x)| < \infty$.

Case 2: $I \neq Ann_r(x)$ and $Ann_r(x)Ann_l(x) = \{0\}$. Then

$$I \rightarrow Ann_r(x) \rightarrow Ann_l(x)$$

and $Ann_l(x) \in ad(I)$. Therefore $Ann_l(x)$ is finite. Since $I \cong R/Ann_l(x)$, $|R| = |I||Ann_l(x)| < \infty$.

Case 3: $I \neq Ann_r(x)$ and $Ann_r(x)Ann_l(x) \neq \{0\}$. Then

$$Ann_r(x) \leftarrow I \rightarrow Ann_r(x)Ann_l(x) \rightarrow (xR)$$

and $(xR), Ann_r(x) \in ad(I)$. Therefore (xR) and $Ann_r(x)$ are finite. Since $(xR) \cong R/Ann_r(x)$, $|R| = |(xR)||Ann_r(x)| < \infty$. \square

Proposition 4.1.7 *Let R be a ring and I be a left principal ideal of R . If $ad(I)$ has ACC on its left and right ideals, then R is left or right Artinian.*

Proof. Let $I = Rx$, for some $x \in R$.

Case 1: $I = \text{Ann}_r(x)$ and $\text{Ann}_r(x)\text{Ann}_l(x) = \{0\}$. Then

$$I \rightarrow \text{Ann}_l(x)$$

and so $\text{Ann}_l(x) \in \text{ad}(I)$. Therefore $\text{Ann}_l(x)$ is left Artinian R -module.

Since $I \cong R/\text{Ann}_l(x)$, we conclude that R is a left Artinian R -module.

Case 2: $I \neq \text{Ann}_r(x)$ and $\text{Ann}_r(x)\text{Ann}_l(x) = \{0\}$. Then

$$I \rightarrow \text{Ann}_r(x) \rightarrow \text{Ann}_l(x)$$

and so $\text{Ann}_l(x) \in \text{ad}(I)$. Therefore $\text{Ann}_l(x)$ is a left Artinian R -module.

Since $I \cong R/\text{Ann}_l(x)$, we conclude that R is a left Artinian R -module.

Case 3: $I \neq \text{Ann}_r(x)$ and $\text{Ann}_r(x)\text{Ann}_l(x) \neq \{0\}$. Then

$$\text{Ann}_r(x) \leftarrow I \rightarrow \text{Ann}_r(x)\text{Ann}_l(x) \rightarrow (xR)$$

and so $(xR), \text{Ann}_r(x) \in \text{ad}(I)$. Therefore (xR) and $\text{Ann}_r(x)$ are right Artinian R -modules. Since $(xR) \cong R/\text{Ann}_r(x)$, we conclude that R is a right Artinian R -module. \square

Theorem 4.1.8 *Let R be a ring which is not a domain. All left ideals of $\text{APO}(R)$ have ACC (resp., DCC) if and only if R is a left Noetherian (resp., Artinian) ring.*

Proof. Suppose that $\text{APO}(R)$ has ACC (resp., DCC) on all its left ideals.

Since R is a non-domain ring, there exist $x, y \in Z(R)^*$ such that $xy = 0$ or $yx = 0$. Without loss of generality assume that $xy = 0$ so $Ann_r(x) \neq 0$. Since for every left ideal $I \subseteq (Rx)$, $IAnn_r(x) = \{0\}$, we conclude that the set $\{I : I \text{ is a left ideal of } R \text{ and } I \subseteq Rx\} \subseteq APO(R)$. Therefore Rx is a left Noetherian (resp., Artinian) R -module since $APO(R)$ has ACC (resp., DCC) on its left ideals. If $Ann_l(x) = \{0\}$, then since $Rx \cong R/Ann_l(x)$, $Rx \cong R$. Thus R is left Noetherian (resp., Artinian) ring.

If $Ann_l(x) \neq \{0\}$, then since $Ann_l(x)(xR) = \{0\}$, $\{I : I \text{ is a left ideal of } R, I \subseteq Ann_l(x)\} \subseteq APO(R)$. So $Ann_l(x)$ is a left Noetherian (resp., Artinian). Note that $Rx \cong R/Ann_l(x)$. Since (Rx) and $Ann_l(x)$ are left Noetherian (resp., Artinian), R is left Noetherian (resp., Artinian).

The converse is clear. □

A directed graph Γ is called a tournament if for every two distinct vertices x and y of Γ exactly one of xy and yx is an edge of Γ . In other words a tournament is a complete graph with exactly one direction assigned to each edge.

Theorem 4.1.9 *Let R be a ring such that $A^2 \neq \{0\}$ for every $A \in IPO(R)^*$ and $A^l(R) \cap A^r(R) \neq \emptyset$. Then $(APO)G(R)$ is not a tournament.*

Proof. Assume $(APO)G(R)$ is a tournament. Since $A^l(R) \cap A^r(R) \neq \emptyset$, there exists $B \in A^l(R) \cap A^r(R)$, that is, there exist distinct $A, C \in IPO(R)^*$ such that $A \rightarrow B \rightarrow C$ is a path in $(APO)G(R)$. If $CA \neq \{0\}$, then $B(CA) = (BC)A = \{0\}$ and $(CA)B = C(AB) = \{0\}$, which is a contradiction. So

$CA = \{0\}$ and therefore $AC \neq \{0\}$ since $(APO)G(R)$ is a tournament. Also, $AC \neq A$ (otherwise $A^2 = (ACAC) = A(CA)C = \{0\}$) and similarly, $AC \neq C$. Let $a, a_1 \in A$ and $c, c_1 \in C$ be arbitrary elements. Then we have $B \rightarrow C \rightarrow ((a - a_1c)R)$ and $(R(c - ac_1)) \rightarrow A \rightarrow B$. As the above $((a - a_1c)R)B = \{0\}$ and $B(R(c - ac_1)) = \{0\}$. Let $b \in B$ be an arbitrary element. Then $-a_1cb = a_1b - a_1cb \in ((a - a_1c)R)B = \{0\}$ and $-bac_1 = bc_1 - bac_1 \in B(R(c - ac_1)) = \{0\}$. Therefore, $ACB = \{0\}$ and $BAC = \{0\}$. Thus both $AC \rightarrow B$ and $B \rightarrow AC$ are edges of $(APO)G(R)$. This is a contradiction, hence, $(APO)G(R)$ cannot be a tournament. \square

4.2 Undirected Annihilating-Product-One Side-Ideal graph

Let R be a ring. We define an undirected graph $\overline{(APO)}G(R)$ with vertices $APO(R)^*$, where distinct vertices A and B are adjacent if and only if either $AB = \{0\}$ or $BA = \{0\}$. The only difference between $(APO)G(R)$ and $\overline{(APO)}G(R)$ is that the former is a directed graph and the latter is undirected (that is, these graphs share the same vertices and the same edges if directions on the edges are ignored). If R is a commutative ring, this definition agrees with the previous definition of the annihilating-ideal graph.

We now show that for a ring R , $\overline{(APO)}G(R)$ is connected, $0 \leq \text{diam}(\overline{(APO)}G(R)) \leq 3$ and $\text{gr}(\overline{(APO)}G(R)) = 3, 4$ or ∞ .

Theorem 4.2.1 *Let R be a ring. Then $\overline{(APO)}G(R)$ is a connected graph and $\text{diam}(\overline{(APO)}G(R)) \leq 3$.*

Proof. Let A and B be distinct vertices of $\overline{(APO)}G(R)$.

Case 1: $AB = \{0\}$ or $BA = \{0\}$. Then $A - B$ is a path.

Suppose $AB \neq \{0\}$ and $BA \neq \{0\}$.

Case 2: $A^2 = \{0\}$ and $B^2 = \{0\}$. Then $A - AB - B$ is a path.

Case 3: $A^2 = \{0\}$ and $B^2 \neq \{0\}$. Then there is a some $C \in APO(R) \setminus \{A, B, 0\}$ such that either $CB = \{0\}$ or $BC = \{0\}$. If either $AC = \{0\}$ or $CA = \{0\}$, then $A - C - B$ is a path. If $AC \neq \{0\}$ and $CA \neq \{0\}$, then $A - CA - B$ is a path iff $BC = \{0\}$ and $A - AC - B$ is a path if $CB = \{0\}$.

Case 4: $A^2 \neq \{0\}$ and $B^2 = \{0\}$. We can use an argument similar to that of the above case to obtain a path.

Case 5: $A^2 \neq \{0\}$ and $B^2 \neq \{0\}$. Then there exist $C, D \in APO(R) \setminus \{A, B, 0\}$ such that either $CA = \{0\}$ or $AC = \{0\}$ and either $DB = \{0\}$ or $BD = \{0\}$. If $C = D$, then $A - C - B$ is a path. If $C \neq D$ and either $CD = \{0\}$ or $DC = \{0\}$, then $A - C - D - B$ is a path. So suppose $CD \neq \{0\}, DC \neq \{0\}$, and $C \neq D$.

Subcase 5.1: $A - C - B$ is a path if $CB = \{0\}$ or $BC = \{0\}$.

Subcase 5.2: $A - CD - B$ is a path if $AC = \{0\}$ or $DB = \{0\}$.

Subcase 5.3: $A - DC - B$ is a path if $CA = \{0\}$ or $BD = \{0\}$.

Subcase 5.4: $A - CB - D - B$ is a path if $AC = \{0\}$, $BD = \{0\}$ and $CB \neq \{0\}$.

Subcase 5.5: $A - BC - D - B$ is a path if $CA = \{0\}$, $DB = \{0\}$, and

$BC \neq \{0\}$.

Thus $\overline{(APO)}G(R)$ is connected and $\text{diam}(\overline{(APO)}G(R)) \leq 3$. \square

Theorem 4.2.2 *Let R be a ring. If $\overline{(APO)}G(R)$ contains a cycle, then $\text{gr}(\overline{(APO)}G(R)) \leq 4$.*

Proof. Let $A_0 - A_1 - A_2 - \dots - A_{n-1} - A_n - A_0$ be a cycle of shortest length in $\overline{(APO)}G(R)$. Assume that $\text{gr}(\overline{(APO)}G(R)) > 4$, i.e., assume $n \geq 4$.

Case 1: There is some A_j such that $A_j \subseteq I = \text{Ann}_l(A_{j+1}) \cap \text{Ann}_l(A_{j-1})$. Without loss of generality assume that $j = 1$. If there is some $0 \neq A \subseteq I$ such that $A \neq A_1$, then $A_0 - A_1 - A_2 - A - A_0$ is a cycle in $\overline{(APO)}G(R)$. So suppose that $I = A_1$. Then either $A_3A_4 = \{0\}$ or $A_4A_3 = \{0\}$. Note that $A_3A_1 \neq \{0\}$, $A_1A_3 \neq \{0\}$, $A_1A_4 \neq 0$ and $A_4A_1 \neq \{0\}$. Thus $A_3A_1 = A_4A_1 = A_1$ since $I = A_1$ is a left ideal. But then $A_3A_4 = \{0\}$ implies $A_3A_1 = A_3(A_4A_1) = (A_3A_4)A_1 = \{0\}$, and $A_4A_3 = \{0\}$ implies $A_4A_1 = A_4(A_3A_1) = (A_4A_3)A_1 = \{0\}$. This is a contradiction.

Case 2: There is some A_j such that $A_j \subseteq \text{Ann}_r(A_{j+1}) \cap \text{Ann}_r(A_{j-1})$. We arrive at a contradiction by an argument similar to that in case 1.

Case 3: For each j , $A_j \not\subseteq \text{Ann}_r(A_{j+1}) \cap \text{Ann}_r(A_{j-1})$ and A_j is not contained in $\text{Ann}_l(A_{j+1}) \cap \text{Ann}_l(A_{j-1})$. Thus, without loss of generality, we have a path in $(APO)G(R)$ of the form $A_0 - A_1 - A_2 - \dots - A_{n-1} - A_n - A_0$.

Subcase 1: $A_1^2 = A_n^2 = \{0\}$. Note that A_0A_n is not a contain in $\{0, A_0, A_n\}$. Then $A_0 - A_0A_n - A_n - A_0$ is a path in $(APO)G(R)$.

Subcase 2: $A_0^2 = \{0\}$ and $A_n^2 \neq \{0\}$. Note that A_0A_{n-1} is not a member of

$\{0, A_0, A_n\}$. Then $A_0 - A_0A_{n-1} - A_n - A_0$ is a path in $(APO)G(R)$.

Subcase 3: $A_0^2 \neq \{0\}$ and $A_n^2 = \{0\}$. Note that A_1A_n is not a member of $\{0, A_0, A_n\}$. Then $A_0 - A_1A_n - A_n - A_0$ is a path in $(APO)G(R)$.

Subcase 4: $A_0^2 \neq \{0\}$ and $A_n^2 \neq \{0\}$. Note that A_1A_{n-1} is not a member of $\{0, A_0, A_n\}$. Then $A_0 - A_0A_{n-1} - A_n - A_0$ is a path in $(APO)G(R)$.

In each of these subcase we have found a cycle in $\overline{(APO)}G(R)$ of length no greater than 4, this is a contradiction.

Since we have found a contradictions in all possible cases, we must have $gr(\overline{(APO)}G(R)) \leq 4$. \square

4.3 Annihilating-Ideal Graphs for Matrices over Commutative Rings

In this section we want to investigate the annihilating-ideal graph of matrices over a commutative ring.

By Theorem 4.2.1, $diam(\overline{(APO)}G(R)) \leq 3$. In the following theorem it is shown that $diam(\overline{(APO)}G(M_n(R))) \geq 2$, where $n \geq 2$. A natural question is whether $diam(\overline{(APO)}G(M_n(R))) \geq diam(\overline{(APO)}G(R))$. We will show that the answer to this question is affirmative.

Theorem 4.3.1 *Let R be a commutative ring. Then $diam(\overline{(APO)}G(M_n(R))) \geq 2$, where $n \geq 2$.*

Proof. Let

$$A = (M_n(R) \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}) \text{ and } B = (\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} M_n(R)).$$

Since

$$A(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} M_n(R)) = \{0\} \text{ and } (M_n(R) \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}) B = \{0\},$$

we conclude that A and B are vertices in $(\overline{APO})G(M_n(R))$. Note that

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^2 \neq 0 \text{ and } \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in A \cap B,$$

so $AB \neq \{0\}$. Therefore, $\text{diam}((\overline{APO})G(M_n(R))) \geq 2$. \square

Theorem 4.3.2 *Let R be a commutative ring. Then $\text{diam}((\overline{APO})G(M_n(R))) \geq \text{diam}(\mathbb{A}\mathbb{G}(R))$.*

Proof. By [15, Theorem 2.1], $\text{diam}(\mathbb{A}\mathbb{G}(R)) \leq 3$.

Case 1: $\text{diam}(\mathbb{A}\mathbb{G}(R)) \leq 2$. By Theorem 4.3.1, $\text{diam}(\overline{(APO)}G(M_n(R))) \geq 2$, so $\text{diam}(\overline{(APO)}G(M_n(R))) \geq \text{diam}(\mathbb{A}\mathbb{G}(R))$.

Case 2: $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 3$. Then there exist ideals $I, J, K, L \in \mathbb{A}\mathbb{G}(R)^*$ such that $I - K - L - J$ is the shortest path between I and J . Since I and J are vertices of $\mathbb{A}\mathbb{G}(R)$, $M_n(I)$ and $M_n(J)$ are in $APO(R)^*$. Suppose that $\text{diam}(\overline{(APO)}G(M_n(R))) = 2$, so there exists $\alpha = [a_{ij}] \in M_n(R)$ such that $M_n(I)\alpha = \alpha M_n(J) = 0$. Without loss of generality assume that $a_{11} \neq 0$. For every $a \in I$,

$$\begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \alpha = 0,$$

so $aa_{11} = 0$. Therefore $I(a_{11}R) = (0)$. For every $b \in J$,

$$\alpha \begin{bmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = 0.$$

Therefore $(a_{11}R)J = (0)$. Thus $I - (a_{11}R) - J$ gives shorter path between I and J in $\mathbb{A}\mathbb{G}(R)$, yielding a contradiction. Hence $\text{diam}(\overline{(APO)}G(M_n(R))) \geq 3$.

□

It was proved in Theorem 4.2.2 that $\text{gr}(\overline{(APO)}G(R)) \leq 4$. We now show that $\text{gr}(\overline{(APO)}G(M_n(R))) = 3$, where $n \geq 2$.

Theorem 4.3.3 *Let R be a commutative ring. Then $gr(\overline{(APO)}G(M_n(R))) = 3$, where $n \geq 2$.*

Proof. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $(AM_n(R)A) - (BM_n(R)B) - (CM_n(R)C)$ is a cycle in $\overline{(APO)}G(M_n(R))$, and so $gr(\overline{(APO)}G(M_n(R))) = 3$. \square

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